NONLINEAR ACOUSTIC WAVES IN PERIODIC MEDIA

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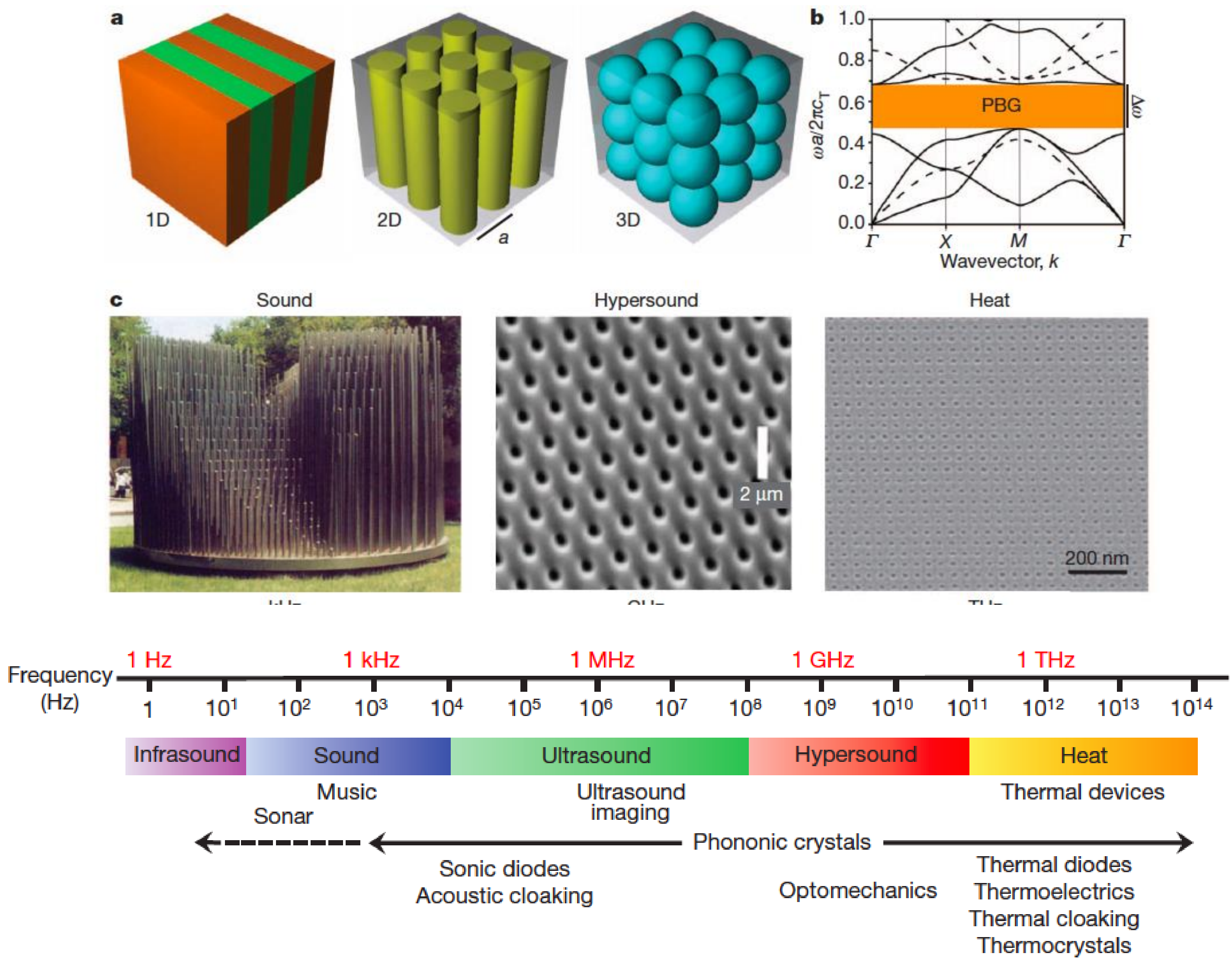
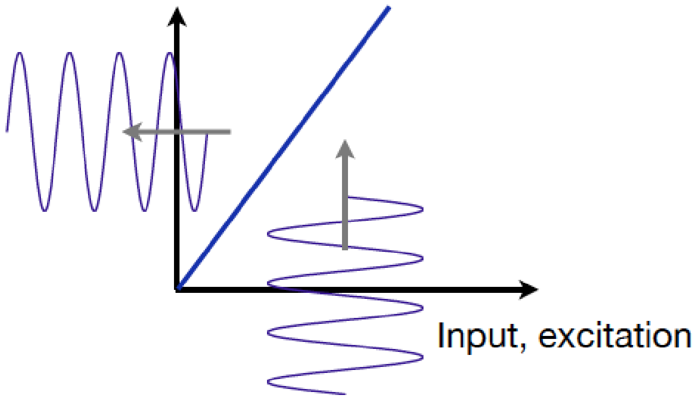
Overview

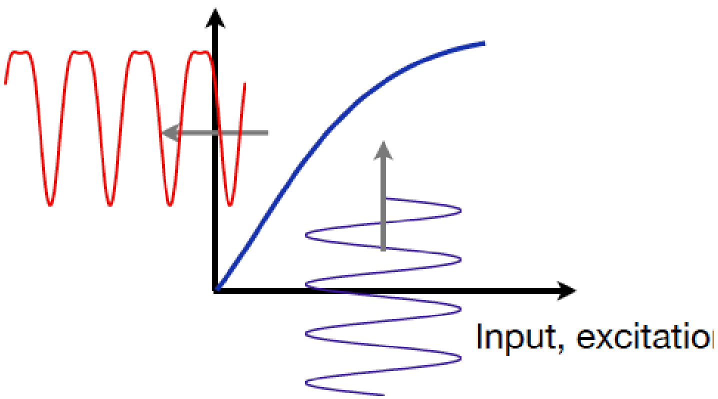
- Introduction:
  - Periodicity and crystals
  - Nonlinearity and Nonlinear Acoustics
- Lattices
  - FPU lattice. Long wavelength limit
- Superlattices or 1D crystals
  - Dispersion relation
  - Harmonic nonlinear effects
  - Solitons in superlattices
- 2D Sonic crystals
  - Self-collimation

Acoustic waves in Crystals



Linear vs nonlinear





Linear system

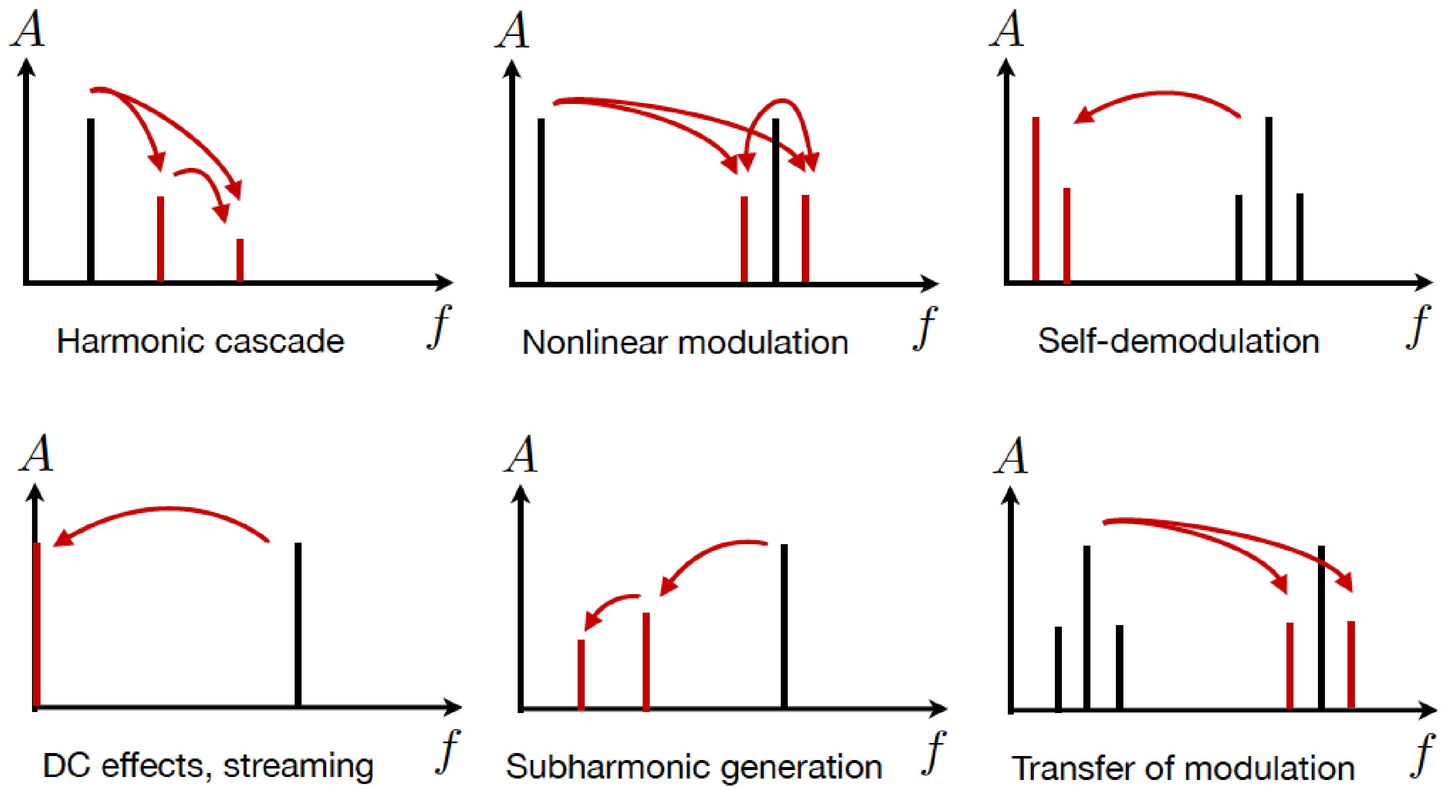
Superposition principle applies
(approximation)

Nonlinear system

Any real systems

➪ Linearity is an approximation (sometimes good) of any intrinsically nonlinear system

Signatures of nonlinearity



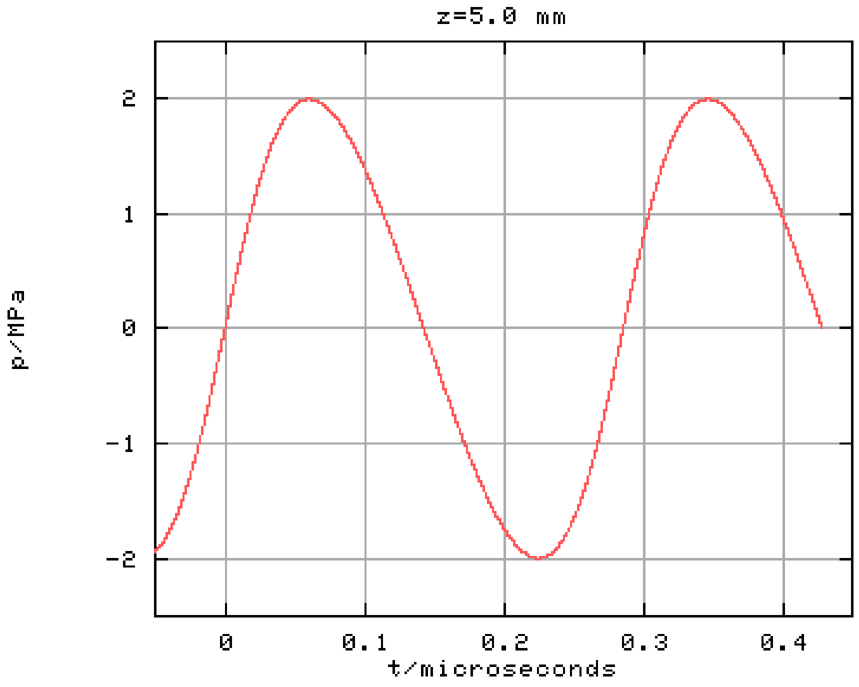
Nonlinear acoustics

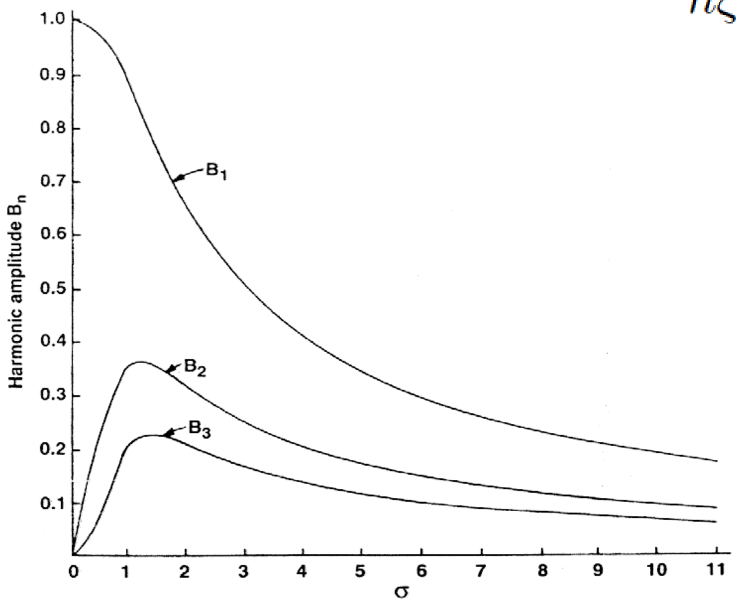
Plane waves in homogeneous medium

Nonlinear and nondispersive

$$p_a = p_0 \sum_{n=1}^{+\infty} B_n(\xi)\sin(n\omega\tau)$$

$$B_n(\xi) = \frac{2J_n(n\xi)}{n\xi}$$





Equations of Nonlinear acoustics

Intense sound waves are accurately described (neglecting dissipation) by the continuity-momentum-state equation system

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial x}$$
$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{\partial P}{\partial x}$$
$$P = P(\rho)$$

A quadratic expansion of the state equation leads to

$$p = c_0^2\rho' + \frac{c_0^2}{\rho_0}\frac{B}{2A}\rho'^2$$

B/A is known as the **coefficient of nonlinearity** of fluids

Equations of Nonlinear acoustics

For 1D plane waves propagating in an ideal gas

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{(1 + \frac{\partial u}{\partial x})^{\gamma+1}} \frac{\partial^2 u}{\partial x^2}$$

For small displacements

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \beta \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}$$

$$\beta = 2 + \frac{B}{A}$$

For fluids

$$\beta = 3 + \frac{c_{111}}{\rho_0 c^2}$$

For solids (along particular directions)

Equations of Nonlinear acoustics - solids

Wave motion in solids is governed by the momentum equation

$$\rho_0 \frac{\partial^2 \mathbf{U}}{\partial t^2} = \boldsymbol{\nabla}_a \cdot \mathbf{P}$$

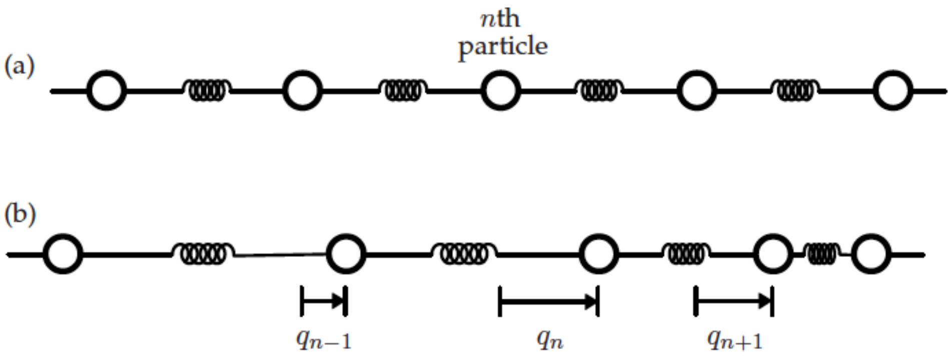
$$P_{ij} = C_{ijkl} \frac{\partial U_k}{\partial a_l} + \frac{1}{2} M_{ijklmn} \frac{\partial U_k}{\partial a_l} \frac{\partial U_m}{\partial a_n} + \frac{1}{3} M_{ijklmnpq} \frac{\partial U_k}{\partial a_l} \frac{\partial U_m}{\partial a_n} \frac{\partial U_p}{\partial a_q} + \cdots$$

$$\rho_0 \frac{\partial^2 U_i}{\partial t^2} = \frac{\partial^2 U_k}{\partial a_j \partial a_l} \left(C_{ijkl} + M_{ijklmn} \frac{\partial U_m}{\partial a_n} + M_{ijklmnpq} \frac{\partial U_m}{\partial a_n} \frac{\partial U_p}{\partial a_q} + \cdots \right) .$$

For a purely longitudinal mode

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \beta \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2}$$

Lattices



The general form (for nearest neighbors coupling)

$$m\frac{\mathrm{d}^2 u_n(t)}{\mathrm{d}t^2} = V'\left(u_{n+1} - u_n\right) - V'\left(u_n - u_{n-1}\right)$$

Many possible interaction potentials

$V'(r) = r^{3/2}$ Hertz

$V'(r) = r^{-2}$ Coulomb

$V'(r) = e^{-r} - 1$ Toda

$\vdots$

Most of then approximate to FPU

$$V'(r) = kr + br^2$$

FPU Lattices

Consider the lattice with quadratic nonlinearity

$$m\frac{d^2u_n}{dt^2} = k\left(u_{n+1} - 2u_n + u_{n-1}\right) - \beta\left(u_{n+1} - u_{n-1}\right)\left(u_{n+1} - 2u_n + u_{n-1}\right)$$

At long wavelengths, the continuum limit applies

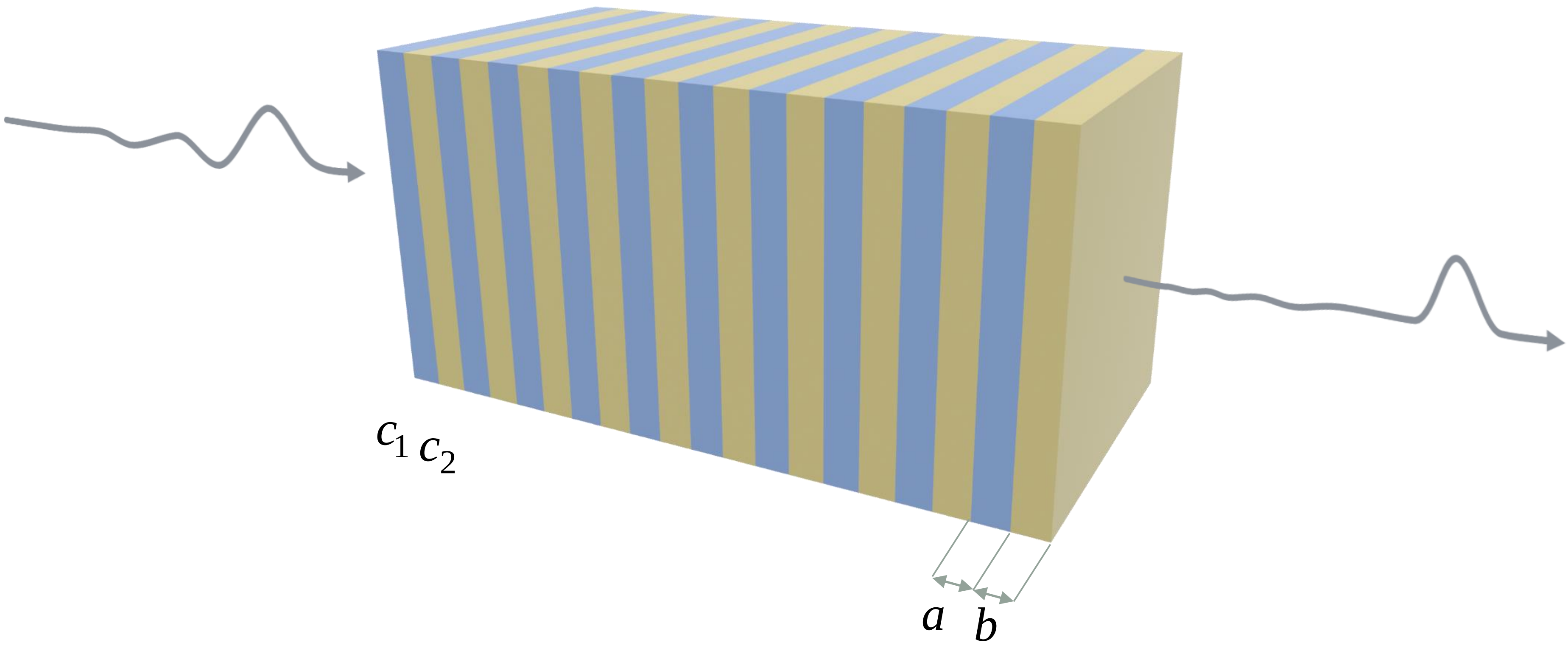
$$u_{n\pm1} = u \pm \frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \ldots$$

The evolution equation reads

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \beta\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}$$

The nonlinear acoustics wave equation!

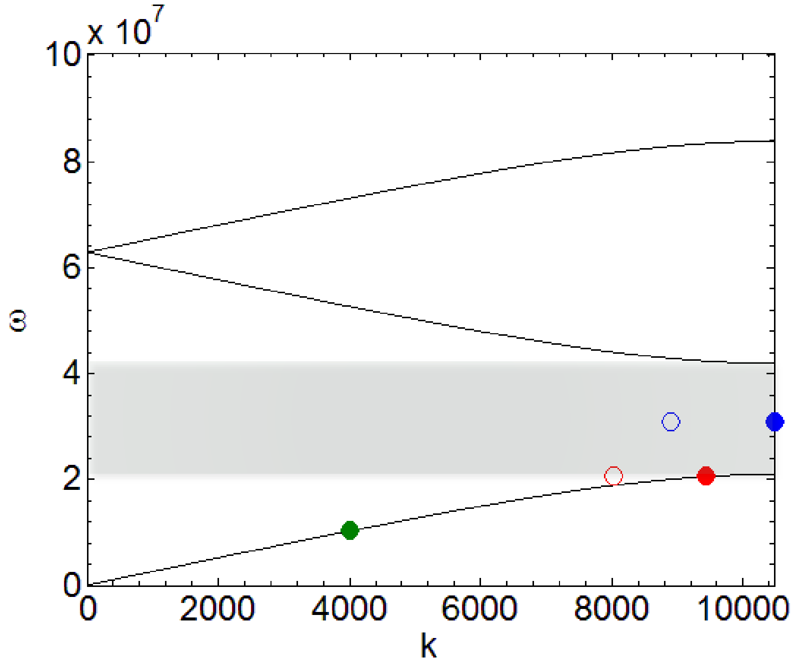
Superlattices (1D crystals)



Dispersion and band structure

The dispersion relation (band structure) is **analytical**

$$\cos(kd) = \cos(k_1 d_1)\cos(k_2 d_2) - \frac{1}{2}\left(\frac{k_1}{k_2} - \frac{k_2}{k_1}\right)\sin(k_1 d_1)\sin(k_2 d_2)$$



One frequency, two modes

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \beta \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}$$

$$u = u_1 + \varepsilon u_2 + \ldots$$

$$\beta = \varepsilon \beta_1$$

$O(\varepsilon^0)$ First harmonic

$$\frac{1}{c^2}\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial^2 u_1}{\partial x^2} = 0$$

$$u_1 = U_1 e^{i(k(\omega)x - \omega t)}$$

$O(\varepsilon^1)$ Second harmonic

$$\frac{1}{c^2}\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial^2 u_2}{\partial x^2} = \beta \frac{\partial u_1}{\partial x}\frac{\partial^2 u_1}{\partial x^2}$$

$$u_2 = U_2^h e^{i(k(2\omega)x - 2\omega t)} + U_2^p e^{i(2k(\omega)x - 2\omega t)}$$

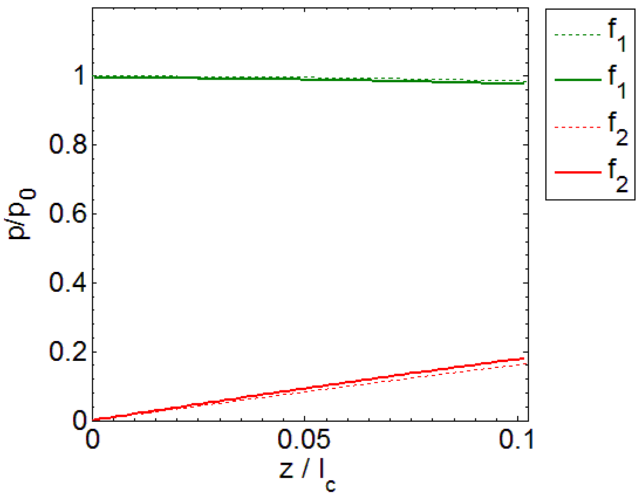
Nonlinear propagation effects in superlattices

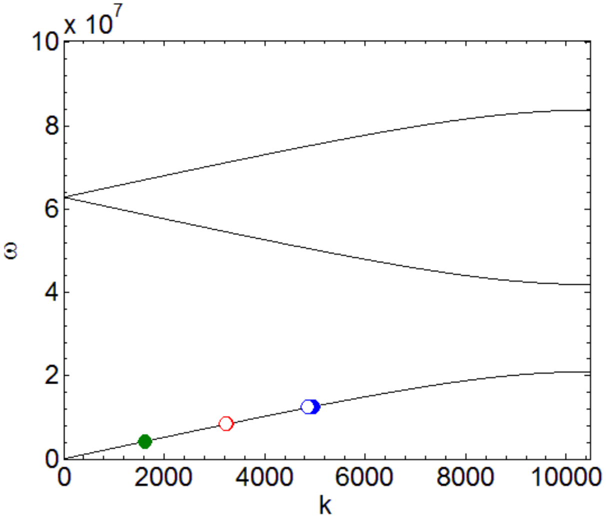
- Asynchronous harmonic generation. Beatings
- Amplitude dependence. Nonlinear dispersion
- Wave propagation in the bandgap
- DC oscillation mode
- Subharmonic generation
- Modulated nonlinearity. Effective cubic nonlinearity
- Solitons

Numerical study by solving constitutive equations for (p, ρ, $\mathbf{v}$) using FDTD method

Harmonic generation – in band case

Low frequency. Weakly dispersive case

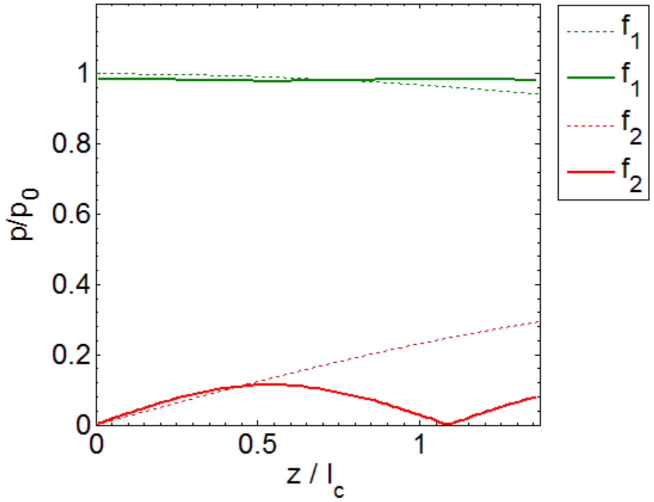


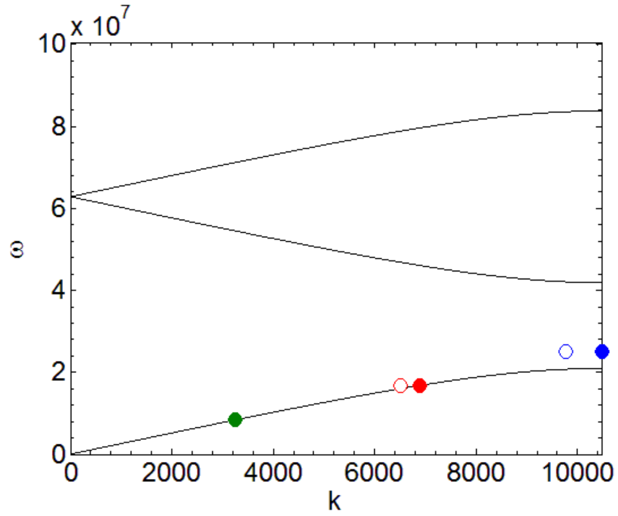


$$|\Delta k| \to 0,\ l_e \to \infty$$

Harmonic generation – in band case

Moderate frequency. Dispersive case

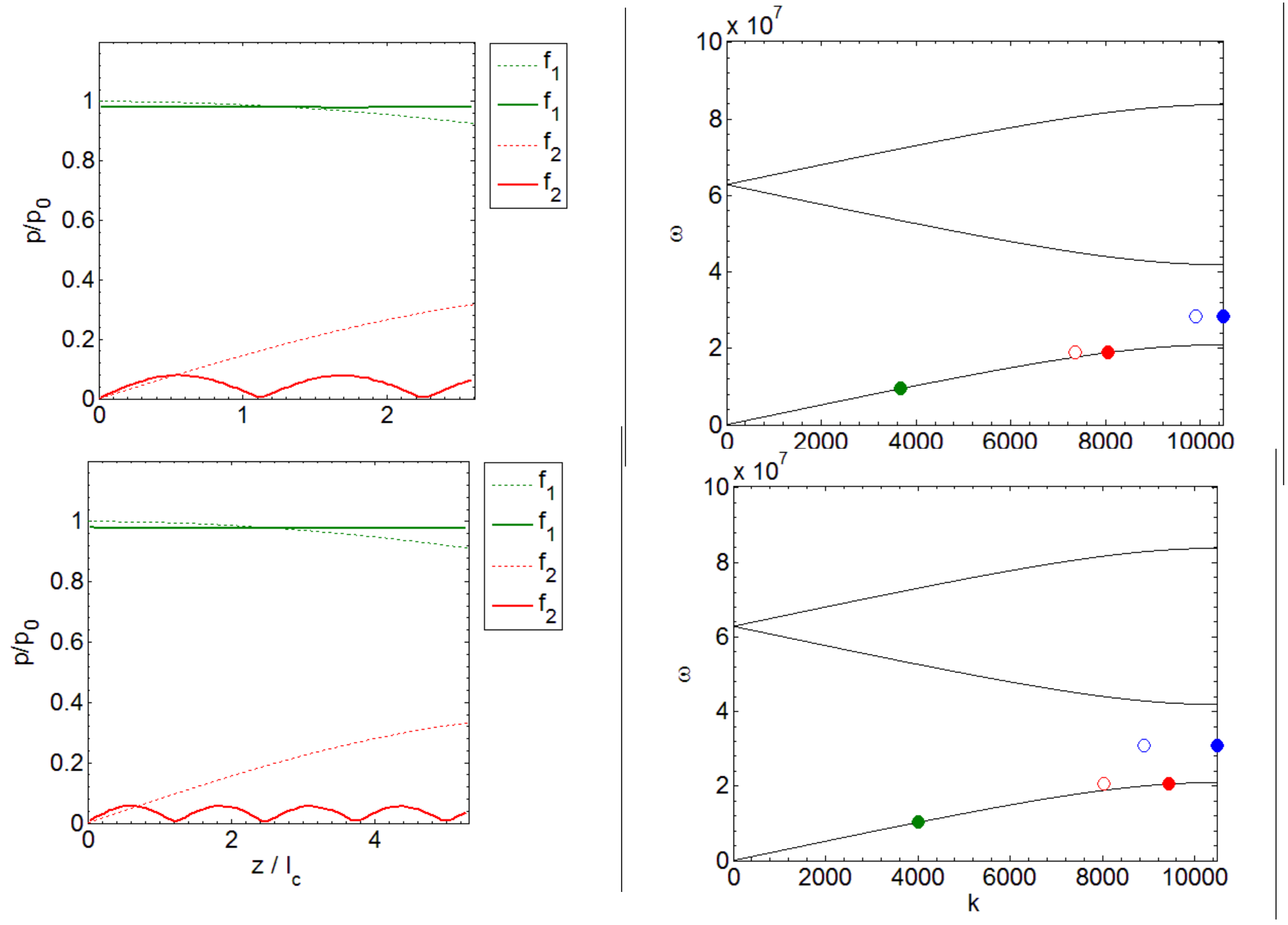




Beatings with a period equal to the coherence length

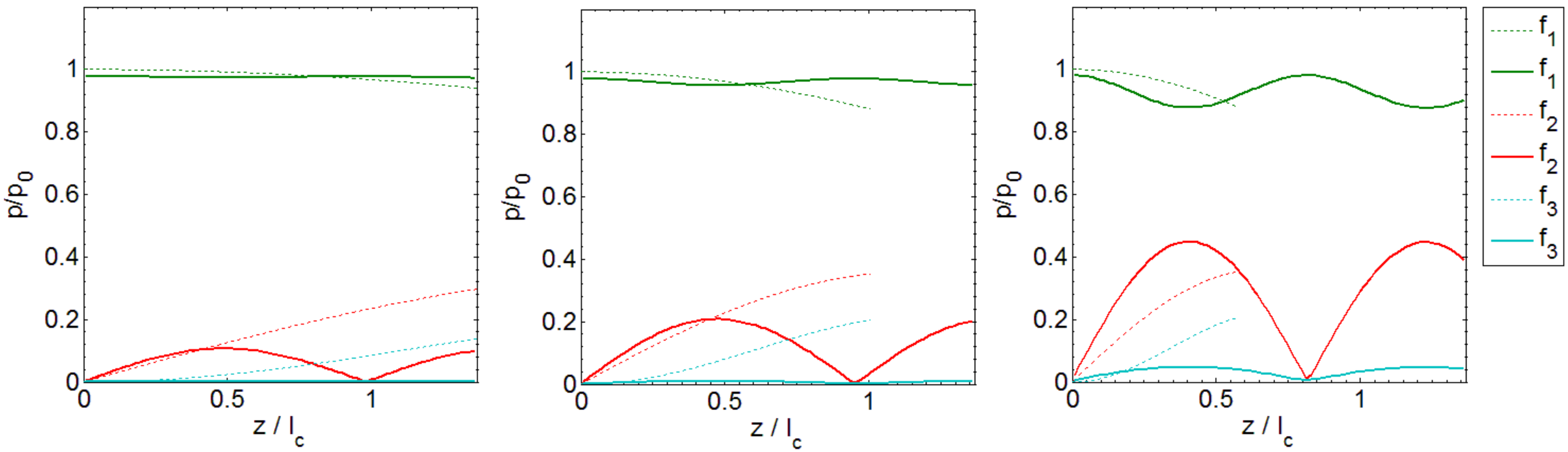
$$l_e = \frac{\pi}{|\Delta k|} = \frac{\pi}{|k(2\omega) - 2k(\omega)|}$$

Harmonic generation - in band case



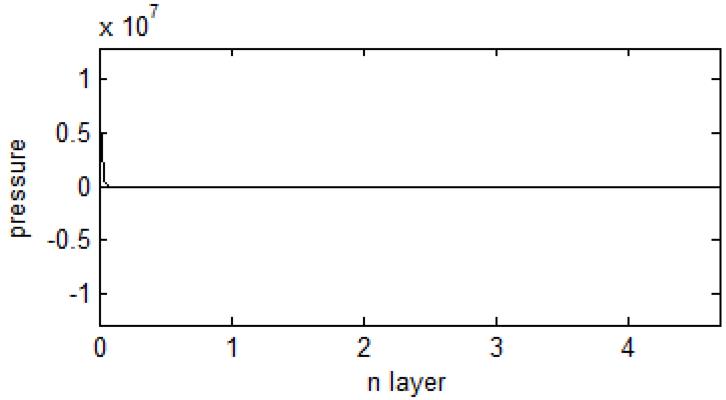
Harmonic generation - nonlinear dispersion

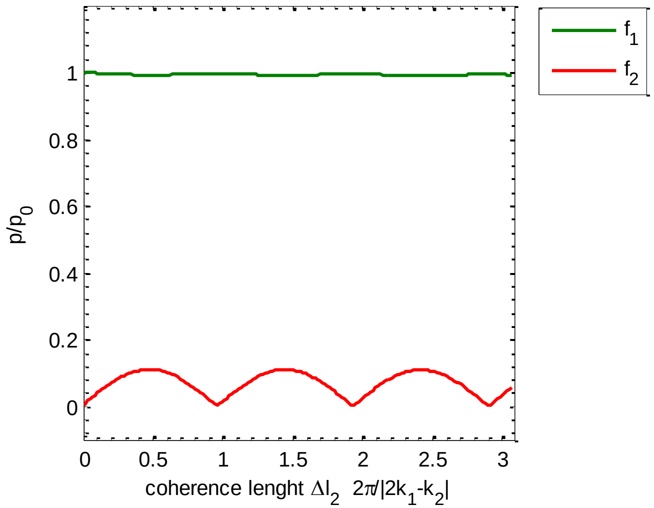
- The beating period depends on the amplitude
- Signature of nonlinear dispersion

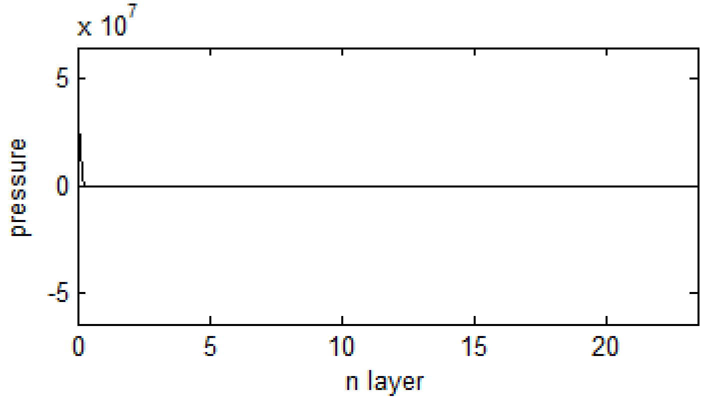


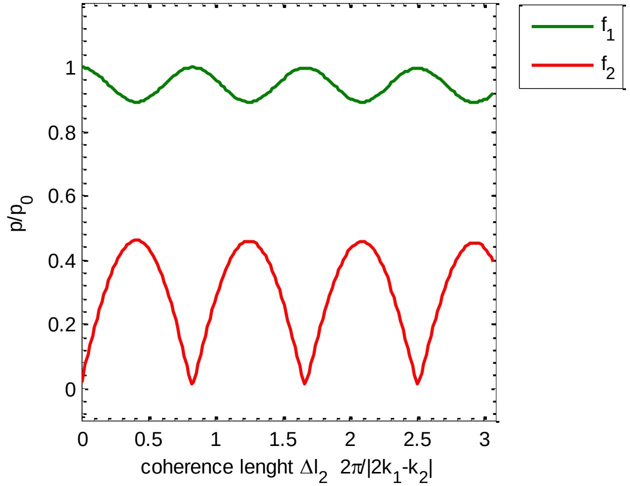
$p_0 = 10$ MPa $\quad$ $p_0 = 50$ MPa $\quad$ $p_0 = 100$ MPa

Harmonic generation – nonlinear dispersion





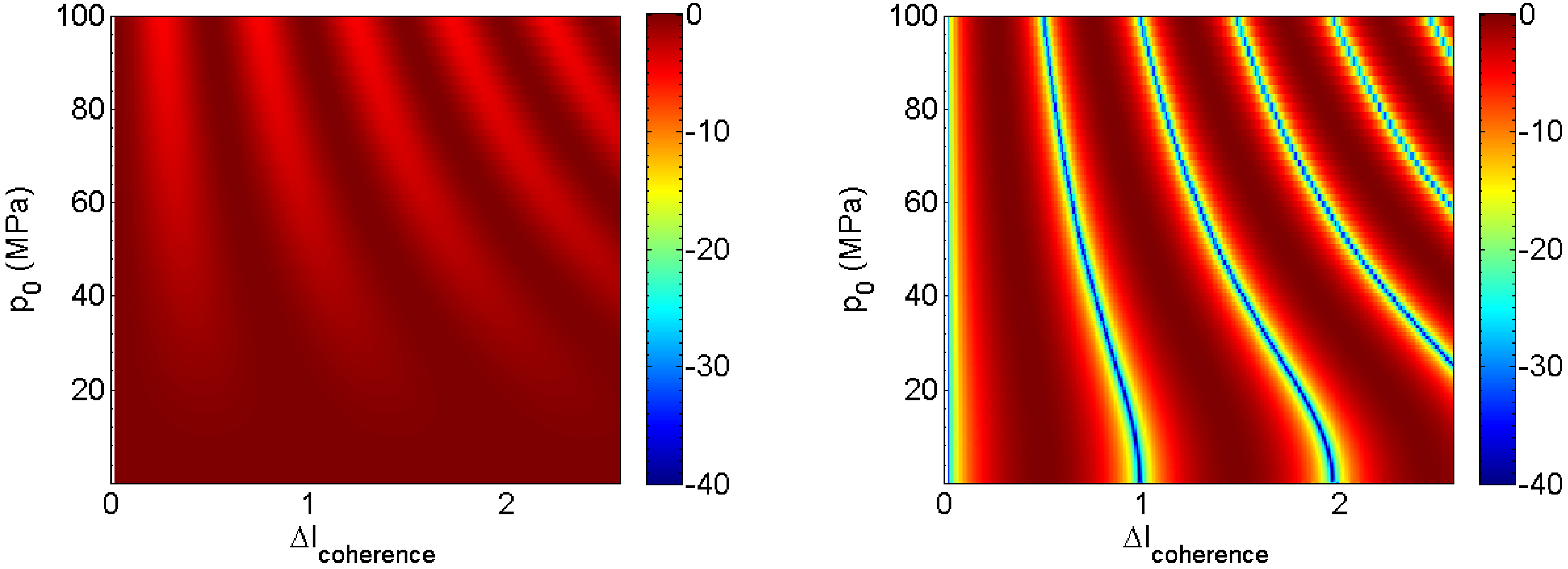




Harmonic generation – nonlinear dispersion

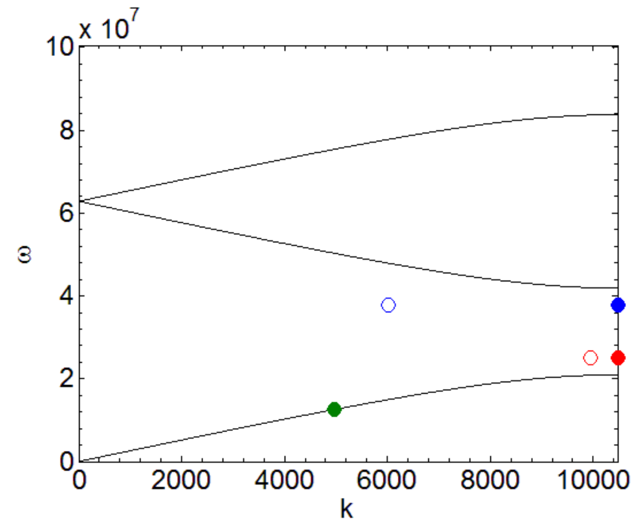
- The full picture

$P_1(p_0, z)$ $P_2(p_0, z)$

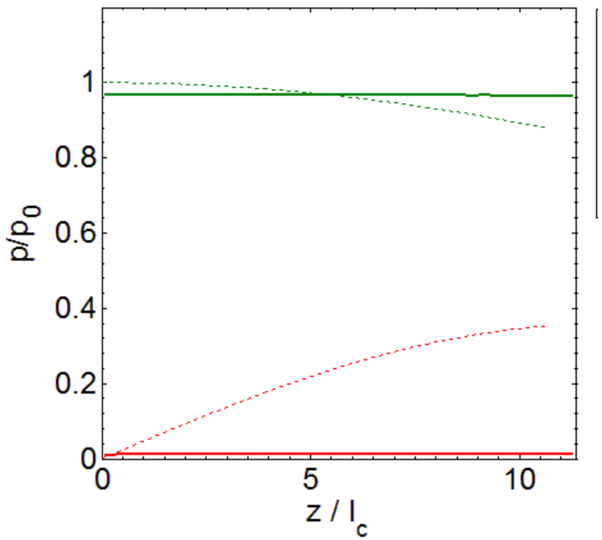


- Dispersion is amplitude dependent
- Computing Δ*k* we can obtain nonlinear variations of dispersion relations

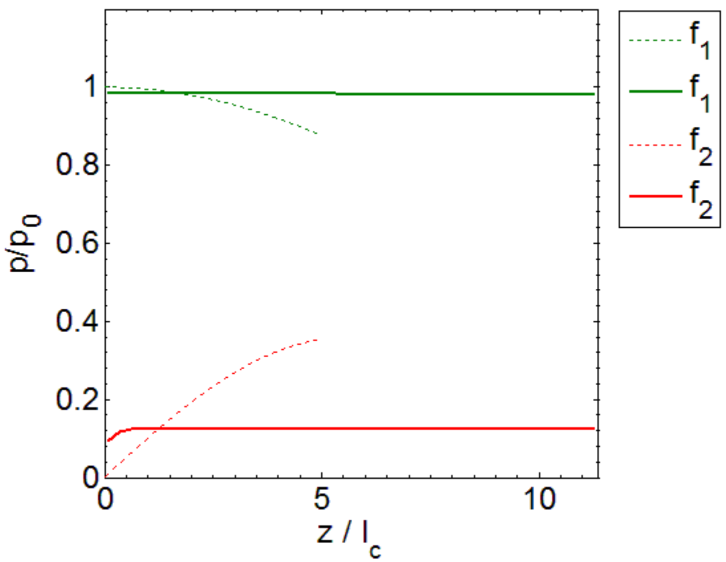
Wave propagation in the bandgap – 2nd harmonic



- 2nd and 3rd harmonics in Band Gap:
- Forced component of 2nd harmonic propagates with finite amplitude



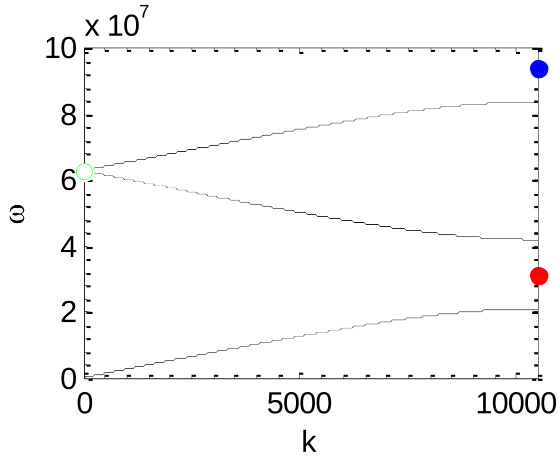
10 MPa

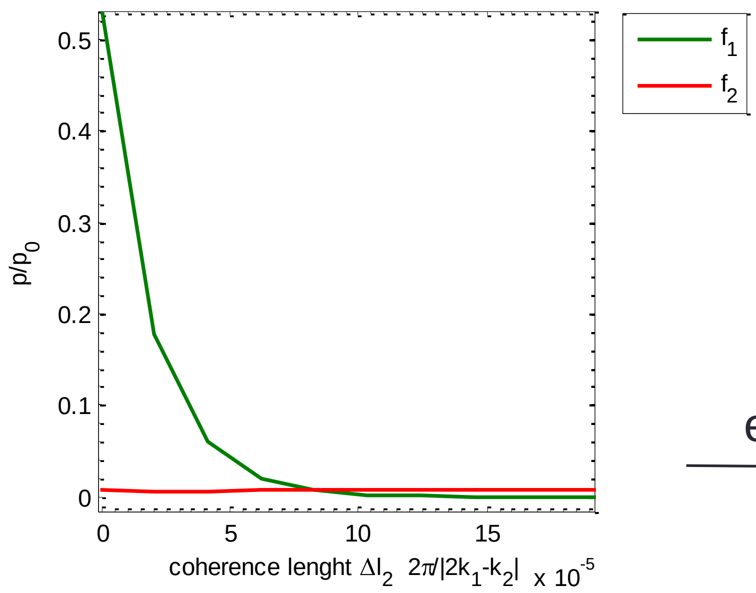


100 MPa

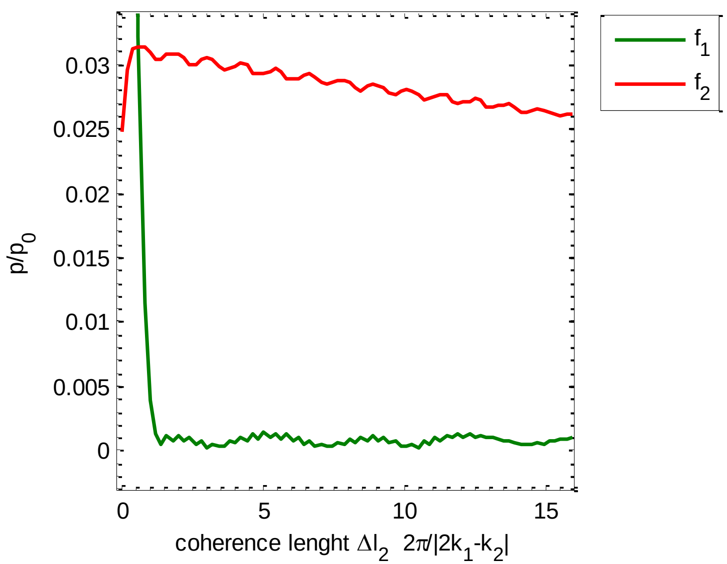
Wave propagation in the bandgap – 1st harmonic

- 1st harmonic in Band Gap
- Phase matched k(2w)=2k(w)
- Evanescent propagation, but…
  - At higher amplitudes 2nd can regenerate 1st



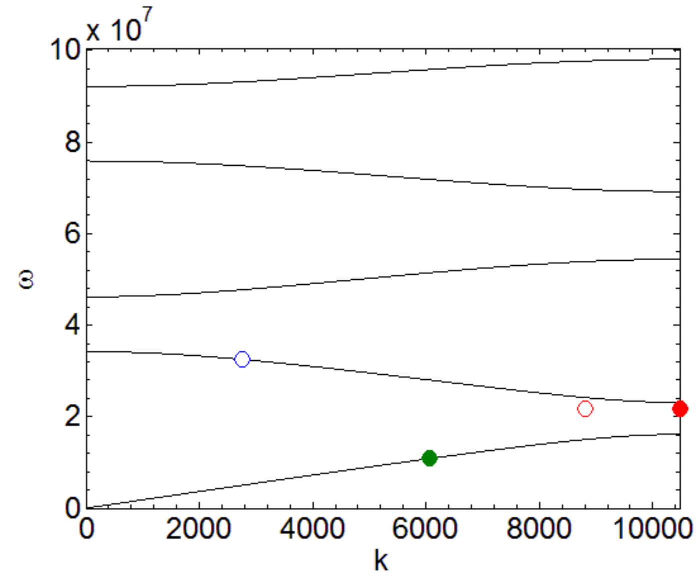


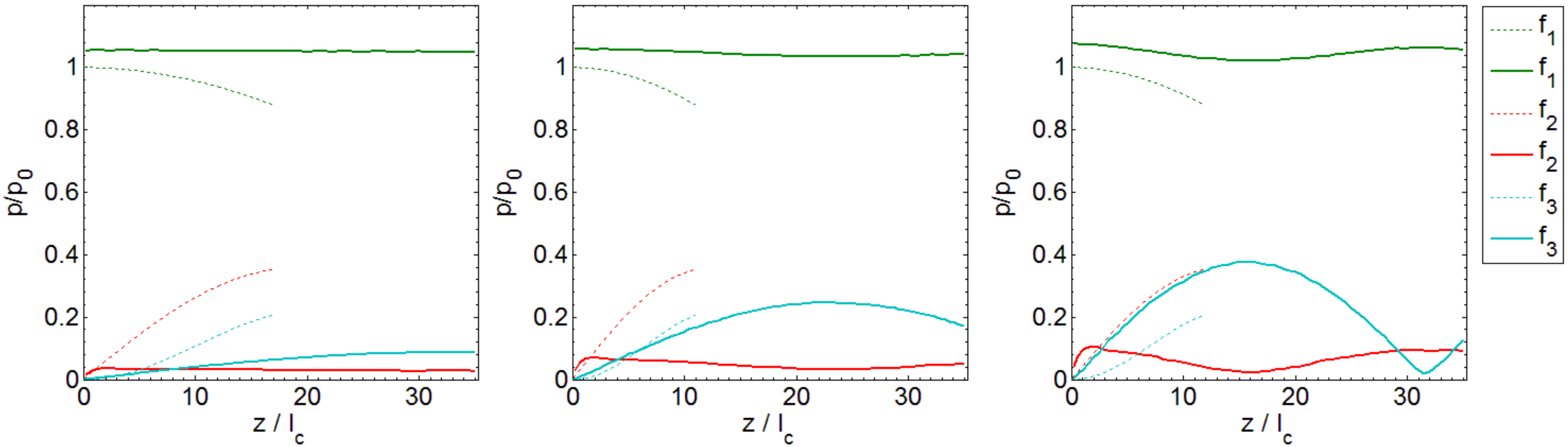
enlarged



3rd Harmonic generation

- 2nd harmonic in Band Gap
- 3rd harmonic phase matched
- The medium behaves as a cubic-like nonlinear material

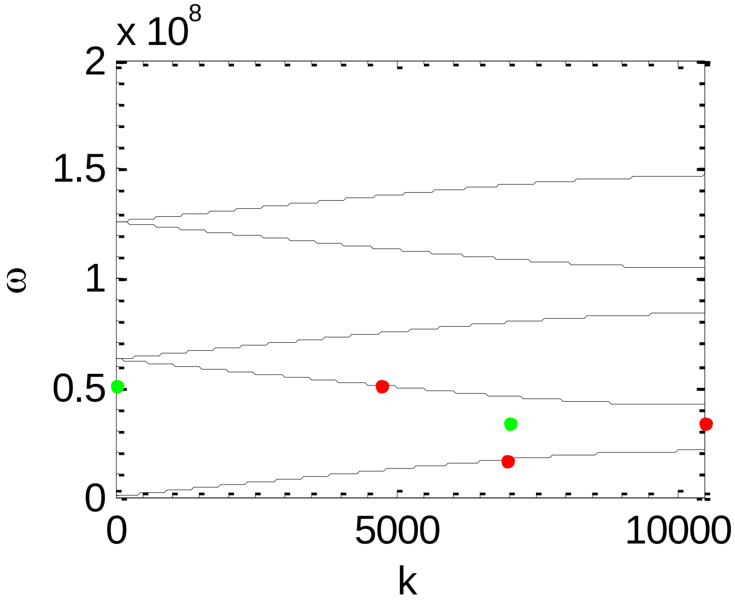


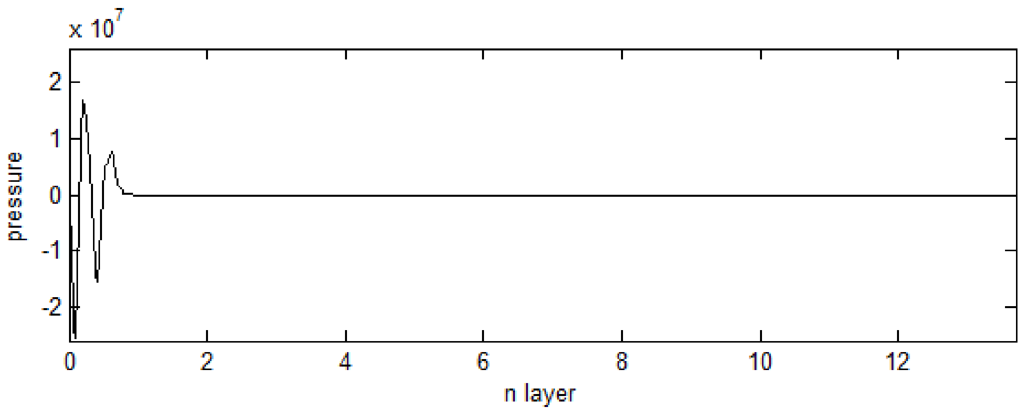


DC oscillation mode

- 2^{nd} harmonic in Band Gap
- 3^{rd} harmonic forced with 3k=0

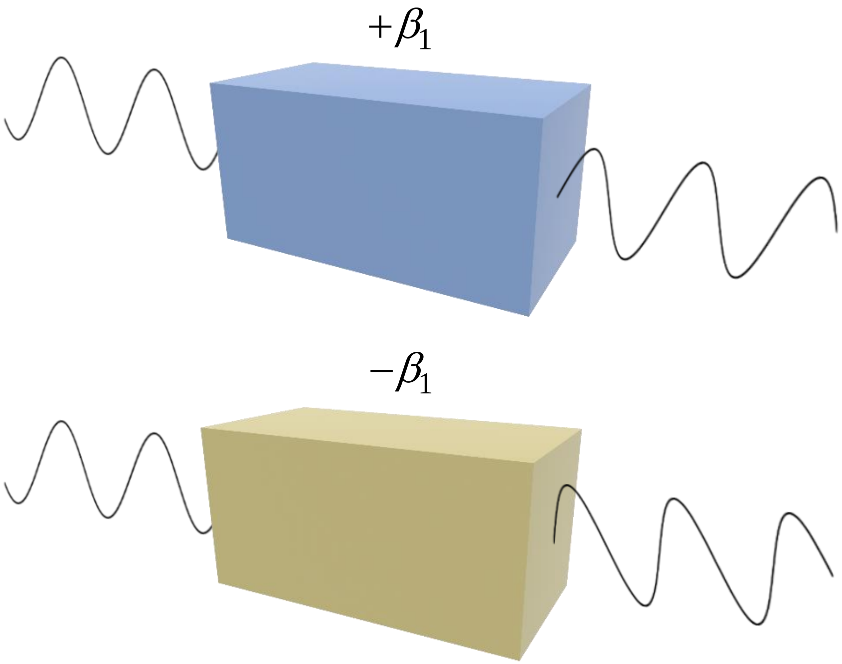
  $3k(\omega)=3k(\omega)-k_B=0$



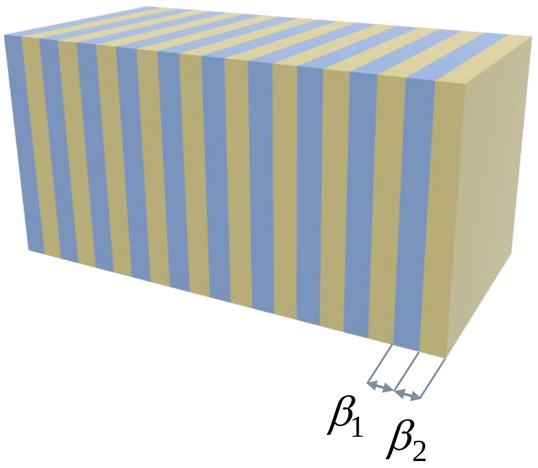


Modulated nonlinearity

- Alternate sign of nonlinearity $c_1 = c_2$ $a = b$

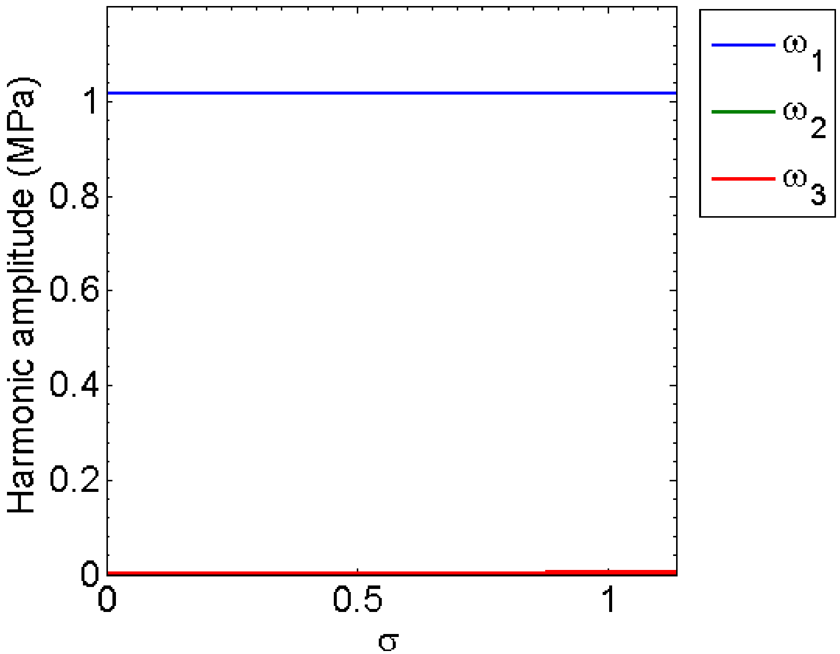


$\beta_2 \approx -\beta_1$



Modulated nonlinearity- Distorsion compensation

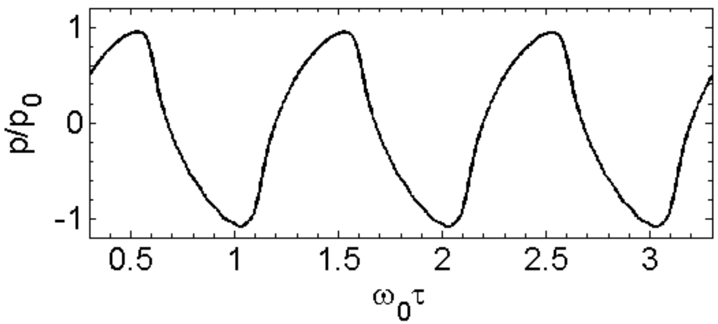
- Nonlinear effects are compensated for $x \sim \sigma$
- Long wavelength $l = 10d, \ d = a + b$

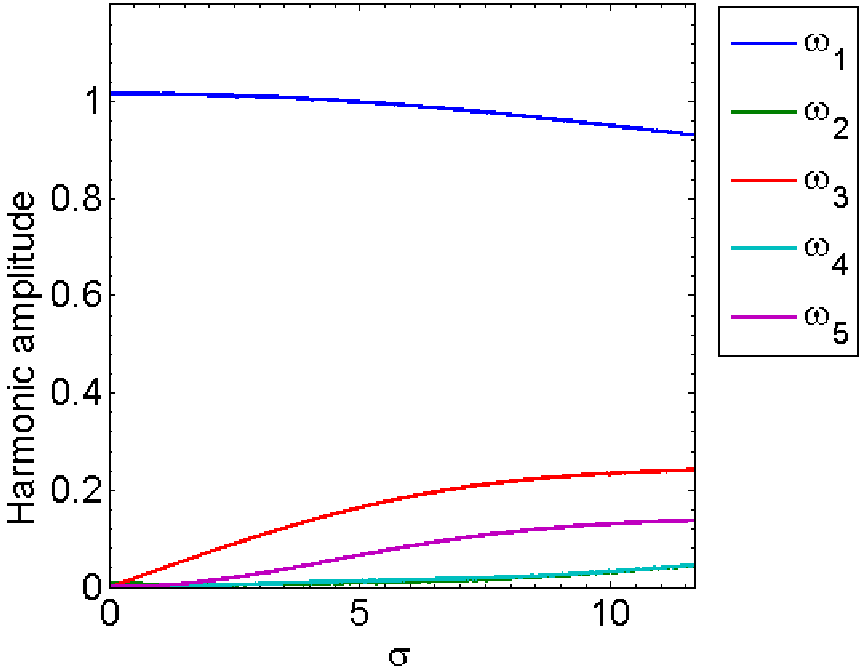


- An extraordinarily linear medium!

Effective cubic nonlinearity

- for $\sigma >> 1$ cubic nonlinear effects appear

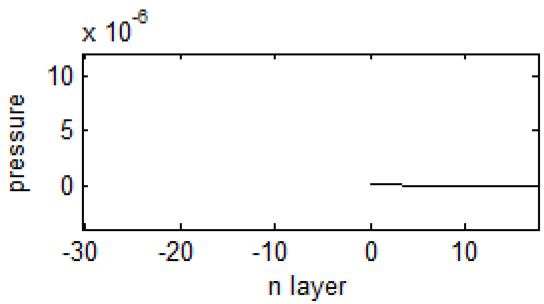




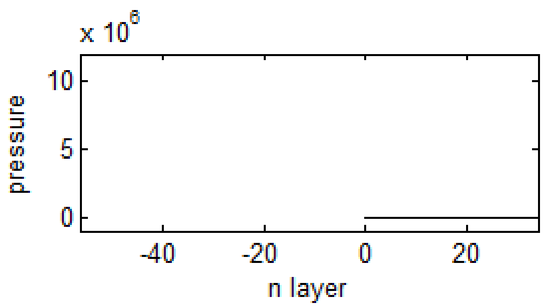
Solitons

Solitons

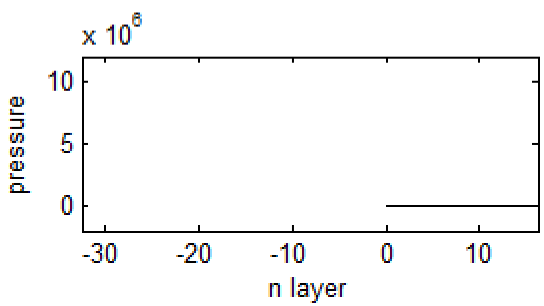
Acoustic pulse propagation under different conditions



Linear and dispersive



Nonlinear and non-dispersive



Nonlinear and dispersive

Solitons in an acoustic superlattice

Progressive nonlinear waves + dispersion

KdV type equation:

$$\frac{\partial u}{\partial t} - c_0 \frac{\partial u}{\partial x} + \mu u \frac{\partial u}{\partial x} + b \frac{\partial^3 u}{\partial x^3} = 0$$

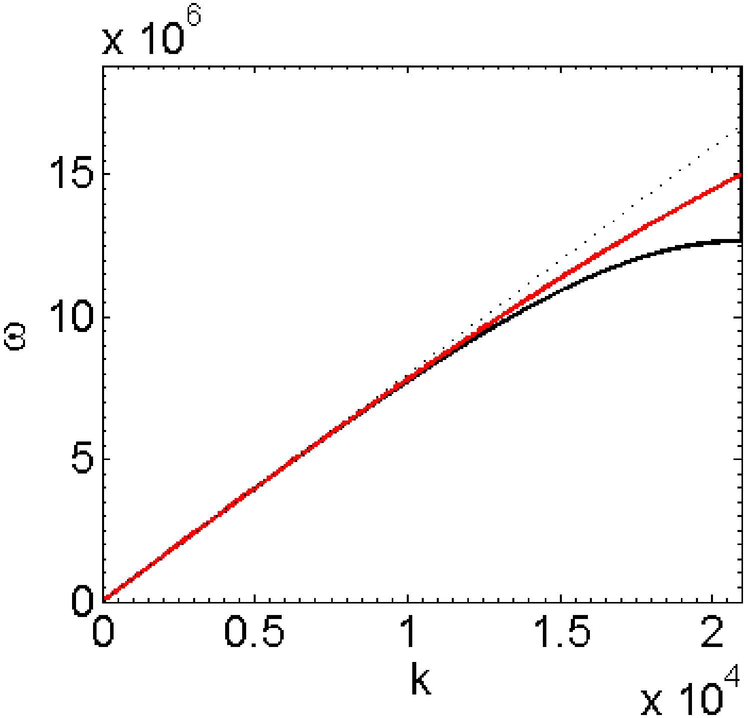
$$\omega(k) = c_0 k - b k^3$$

Effective layered parameters (lower band)

$$c_0 = \sqrt{\frac{(a+b) c_1^2 c_2^2}{b c_1^2 + a c_2^2}}$$

$$b = -\frac{1}{24}(a+b)^2 c_0$$

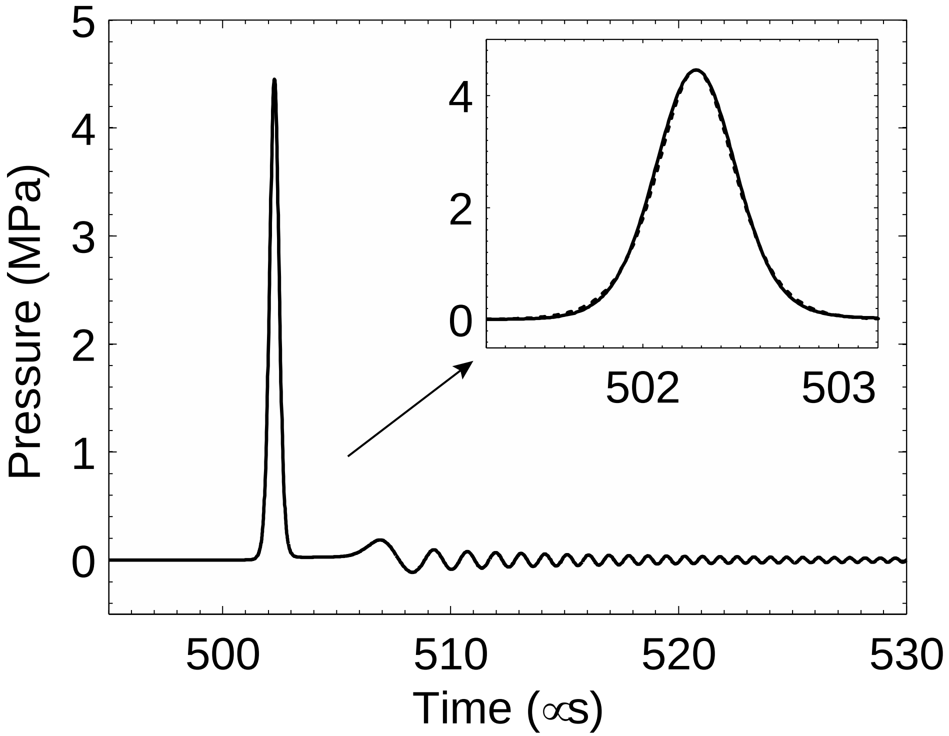
$$\mu = \frac{\beta}{\rho_0 c_0^2}$$



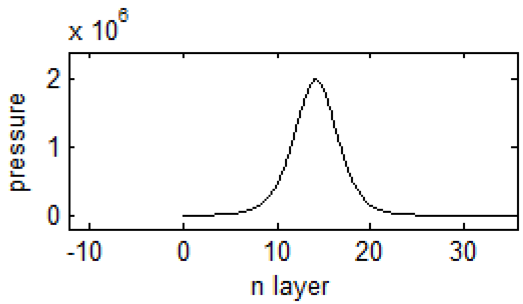
Solitons – analytical solution

Soliton solution for KdV

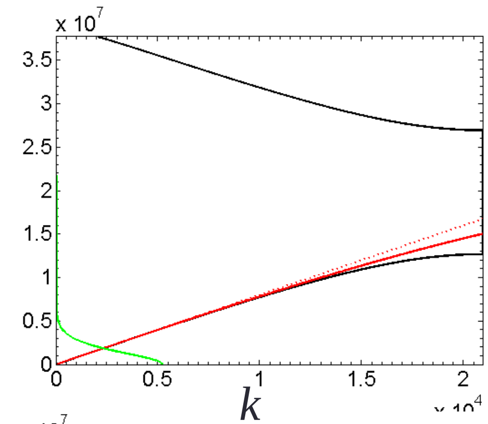
$$u(x,t) = A_0 \operatorname{sech}^2\left(\gamma\left(x - Vt\right)\right) \qquad \gamma = \sqrt{\frac{A_0 \mu}{12b}} \qquad V = c_0 + \frac{\mu A_0}{3}$$

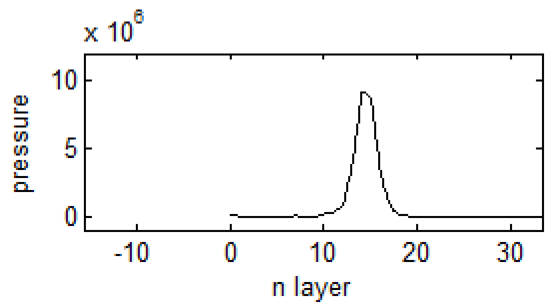


Solitons – amplitude effects

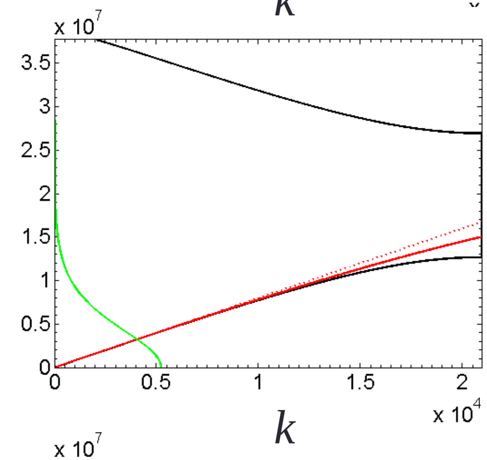


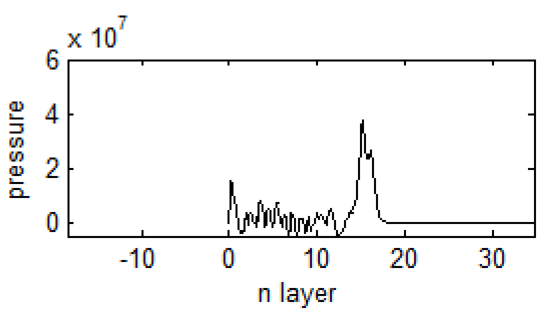
$$p_0 = 2 \text{ MPa}$$



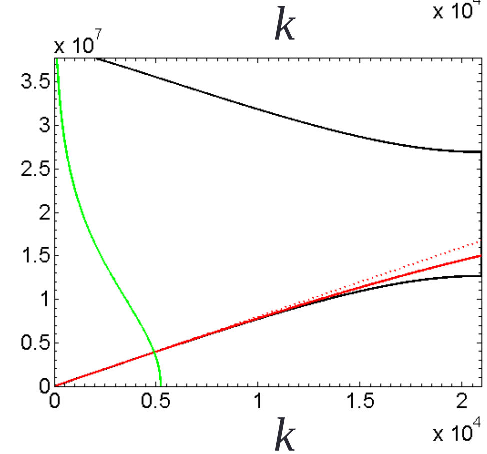


$$p_0 = 10 \text{ MPa}$$



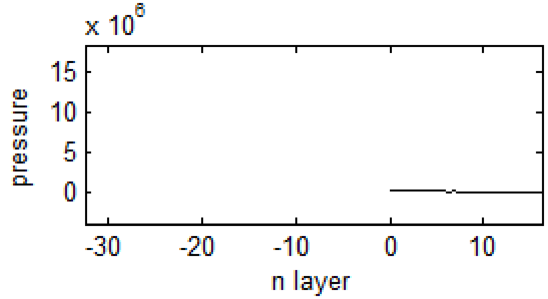


$$p_0 = 50 \text{ MPa}$$

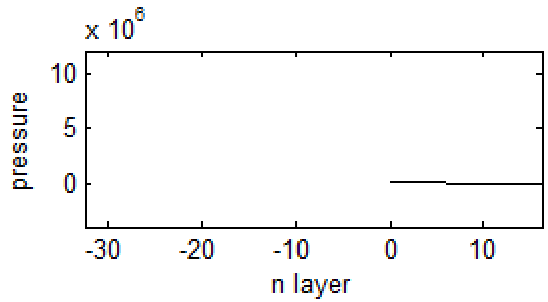


Solitons – width effects

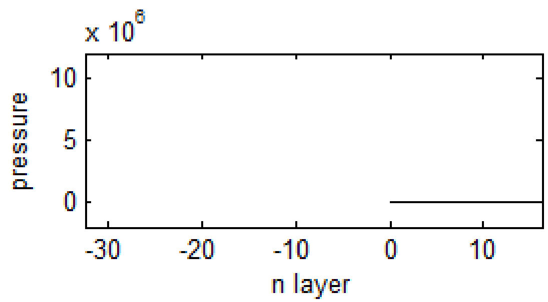
$$g_0 = \sqrt{\frac{A_0 m}{12b}}$$



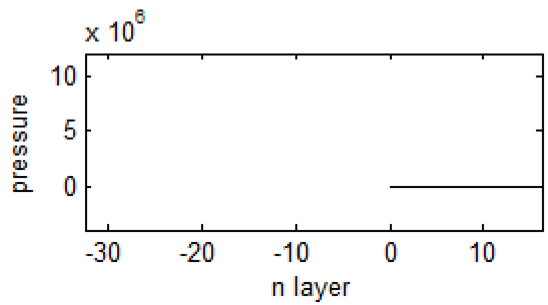
$$g = \frac{1}{5} g_0$$



$$g = \frac{1}{2} g_0$$



$$g = g_0$$



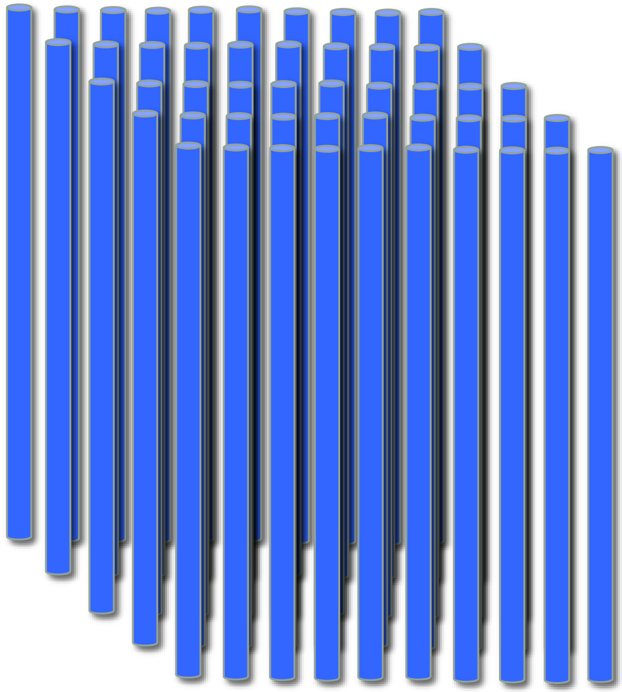
$$g = 2 g_0$$

2D nonlinear sonic crystals

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P$$

$$p = c_0^2 \rho' + \frac{c_0^2}{\rho_0} \frac{B}{2A} \rho'^2$$

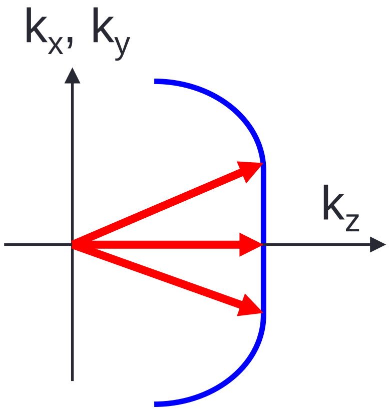


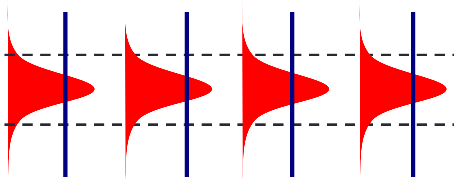
- Scatterers are considered as rigid
- Nonlinearity only in the host medium (fluid)

Self-collimation - review

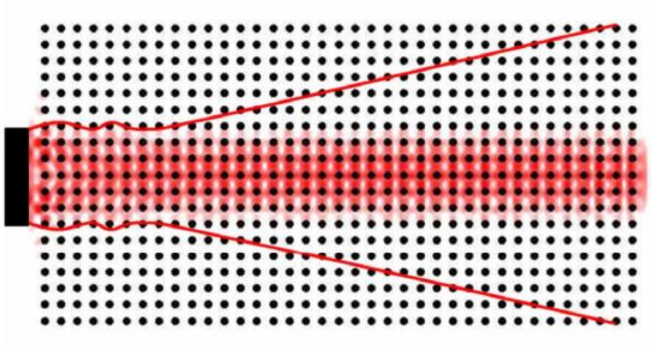
Self-collimation:

- Propagation of PWs is perpendicular to the IFC
- A flat IFC results in an effective zero diffraction

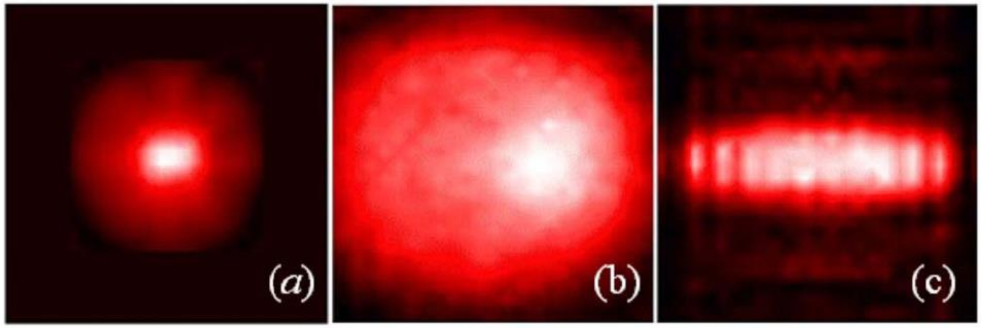




Numerical simulation



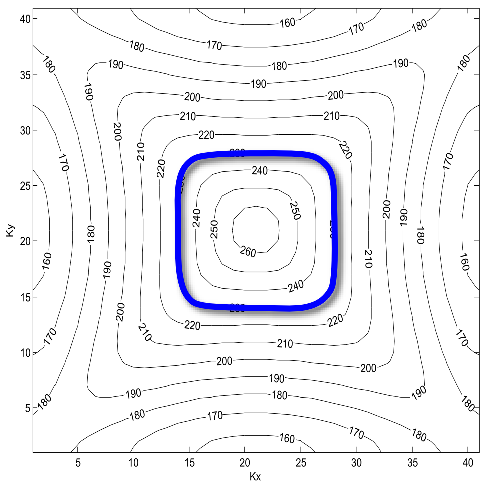
Experimental results



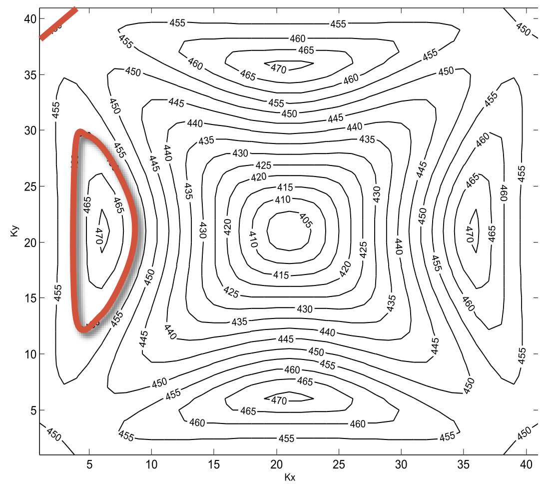
V. Espinosa et al, PRB **76**, 140302R 2007

2D nonlinear sonic crystals

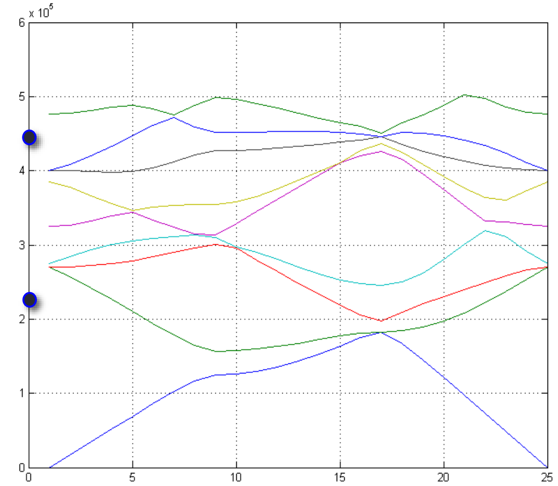
Dispersion curves at self-collimation condition



Second band

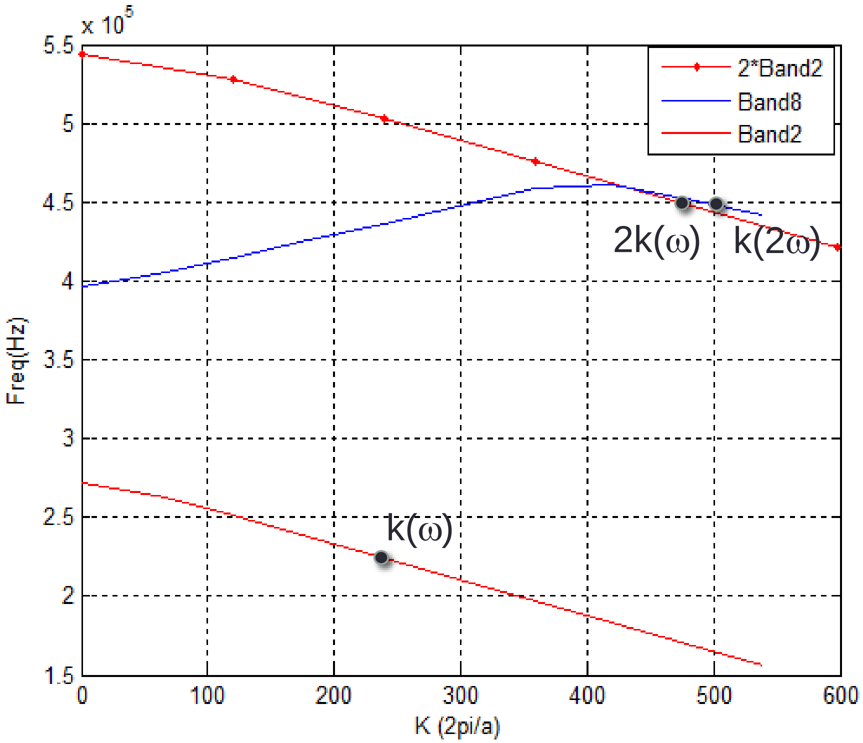


Eigth band

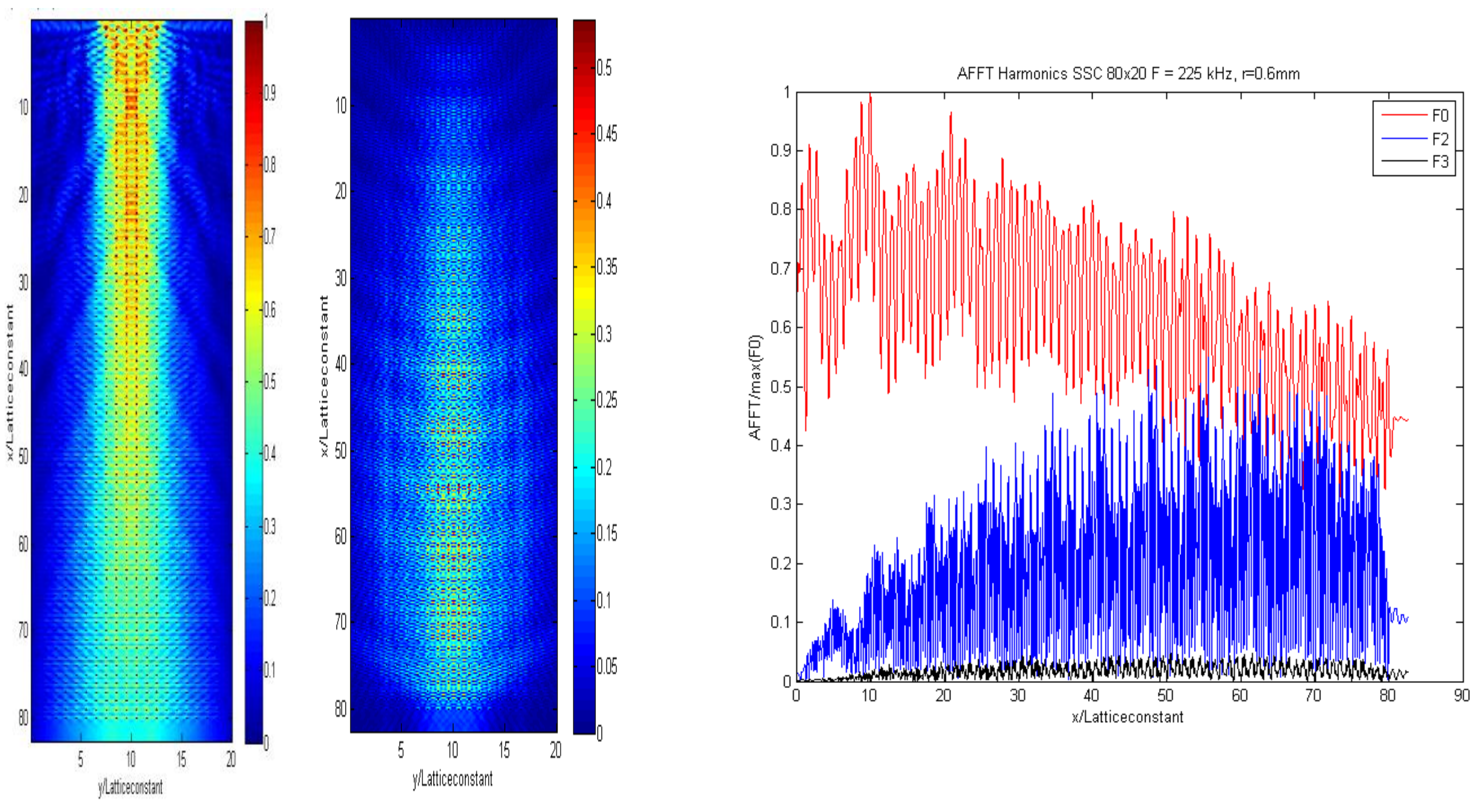


Isofrequency curves and band structure of the crystal:
r = 1mm, a = 5.25mm, host medium: water.

Phase matching of second harmonics

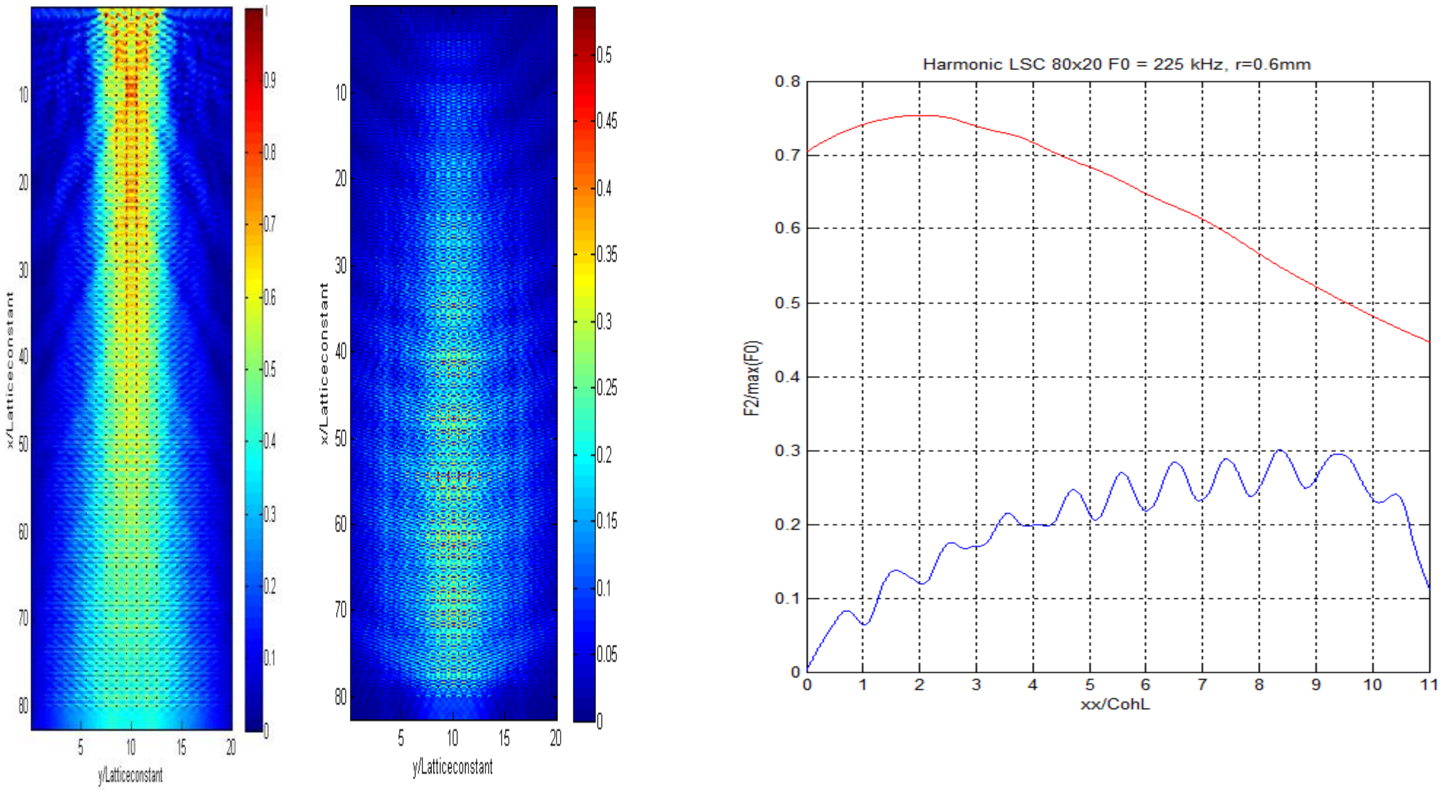


2D nonlinear sonic crystals



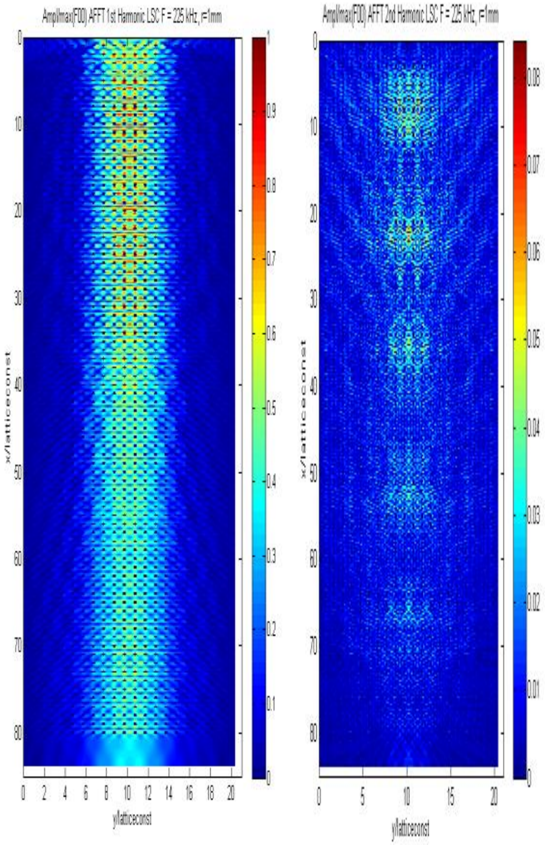
r = 0.6mm, a = 5.25mm, frequency (Fundamental) = 225 kHz, P0 = 1.5MPa, circular scatterers, Transducer diameter Ra = 35mm

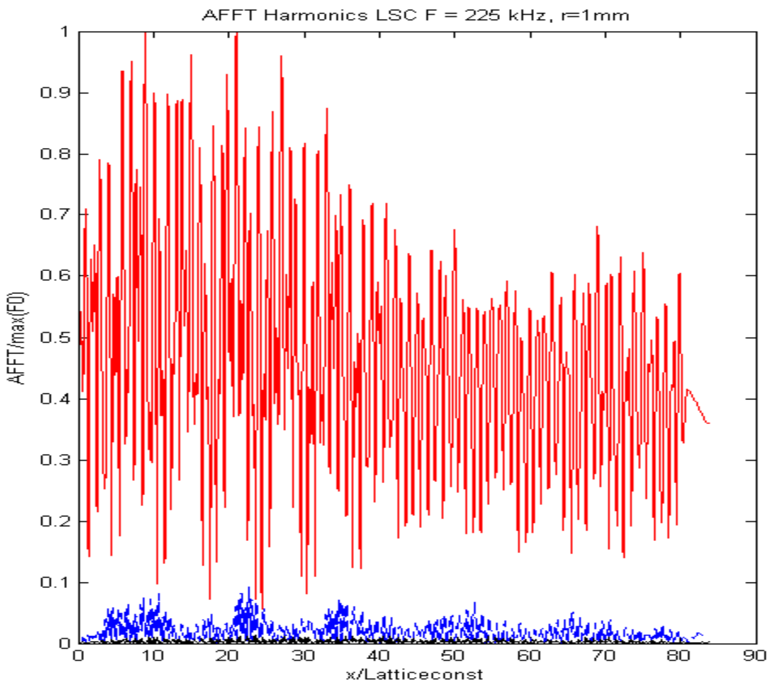
2D nonlinear sonic crystals



r = 0.6mm, a = 5.25mm, frequency (Fundamental) = 225 kHz, P0 = 1.5MPa, circular scatterers, Transducer diameter Ra = 35mm

2D nonlinear sonic crystals





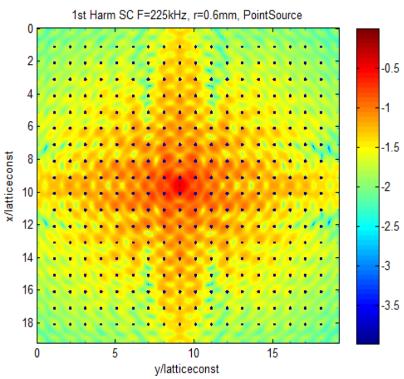
r = 1 mm, a = 5.25 mm, frequency (Fundamental harmonic) = 225 kHz, P0 = 1.5MPa, circular scatterers, Transducer diameter Ra = yy/3 = 35mm

2D nonlinear sonic crystals – point source

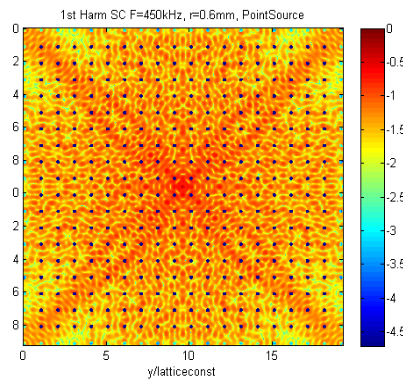
Point source at the center of the crystal

r=0.6 mm

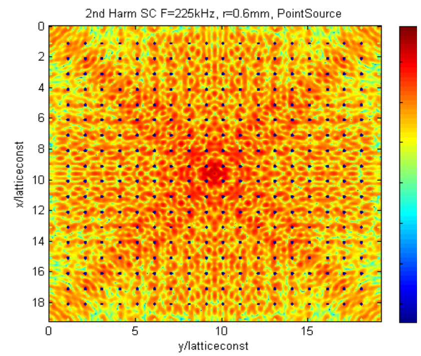
1st harmonic



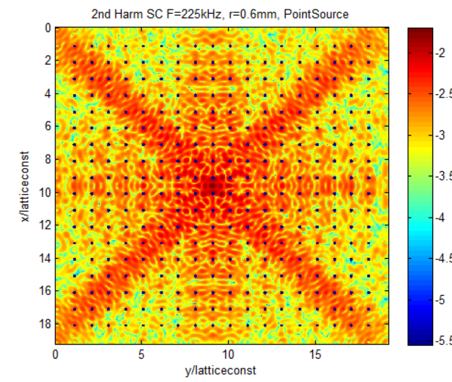
2nd harmonic
Linear



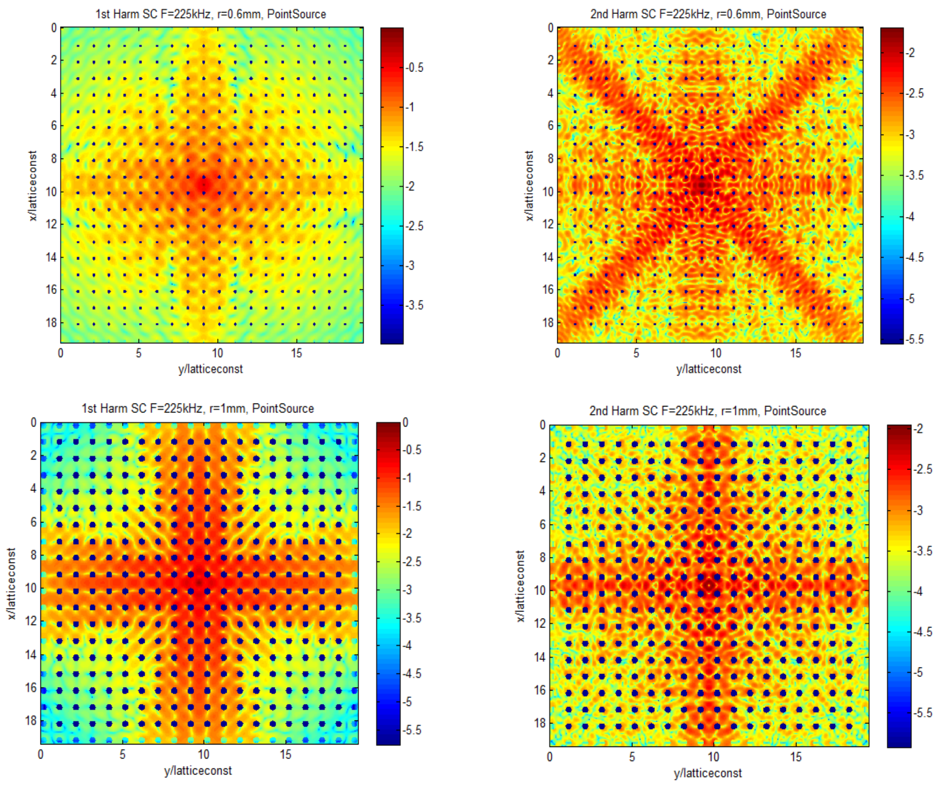
2nd harmonic
Nonlinear, weak



2nd harmonic
Nonlinear, strong



2D nonlinear sonic crystals

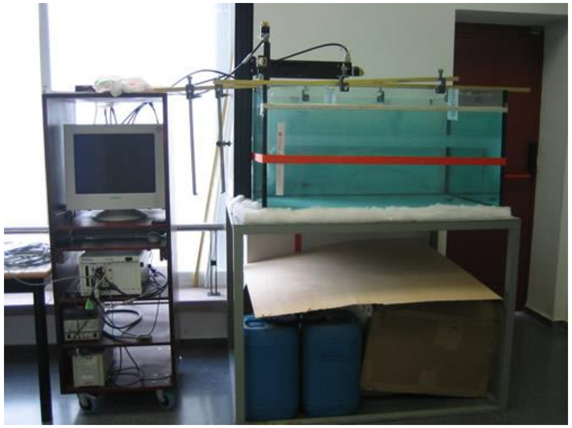


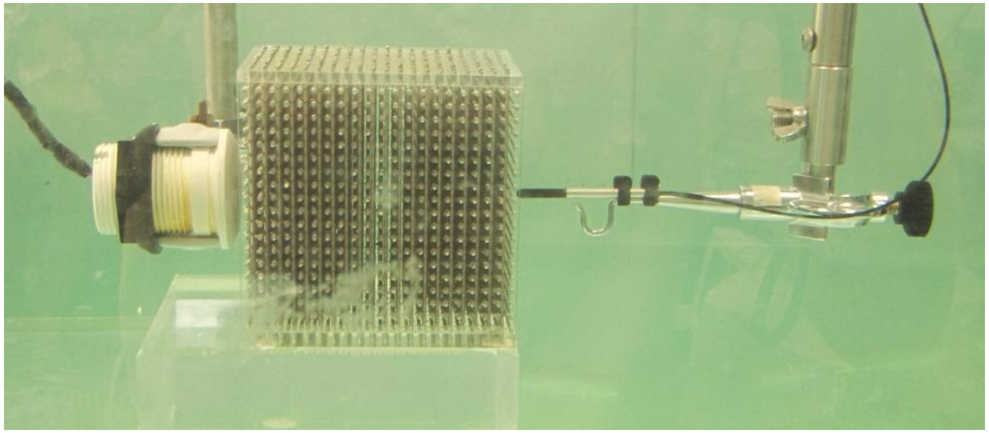
r=0.6mm

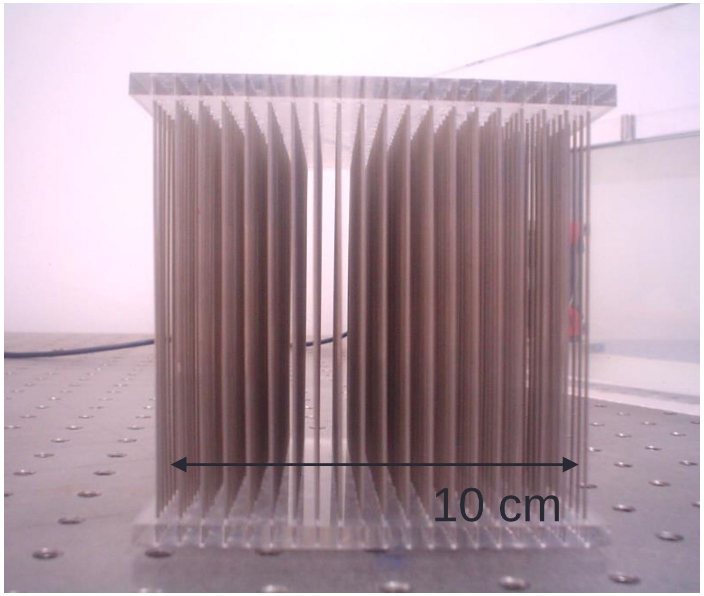
r=1mm

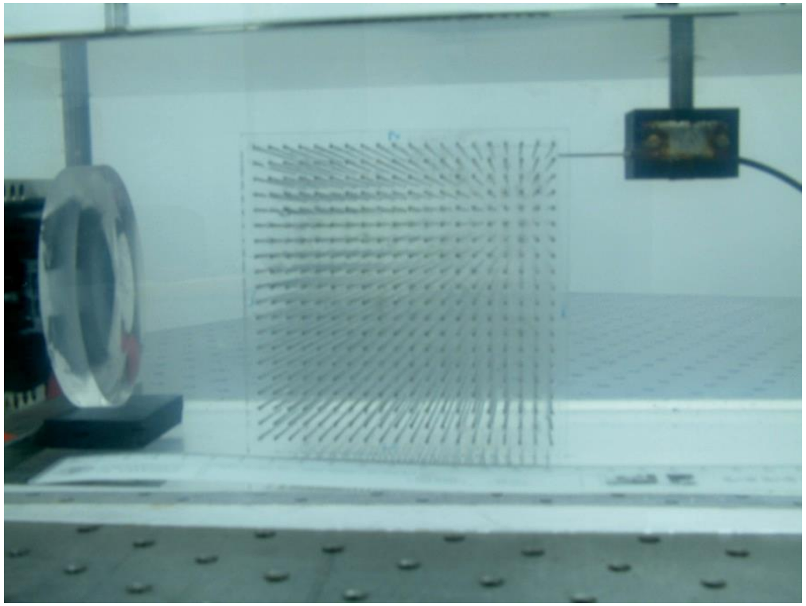
- Increasing filling factor modifies propagation directions (dispersion relations change)
- Since dispersion relations change with nonlinearity, it is possible to select the directions by varying the amplitude (nonlinear spatial filter)

2D nonlinear sonic crystals - experiments

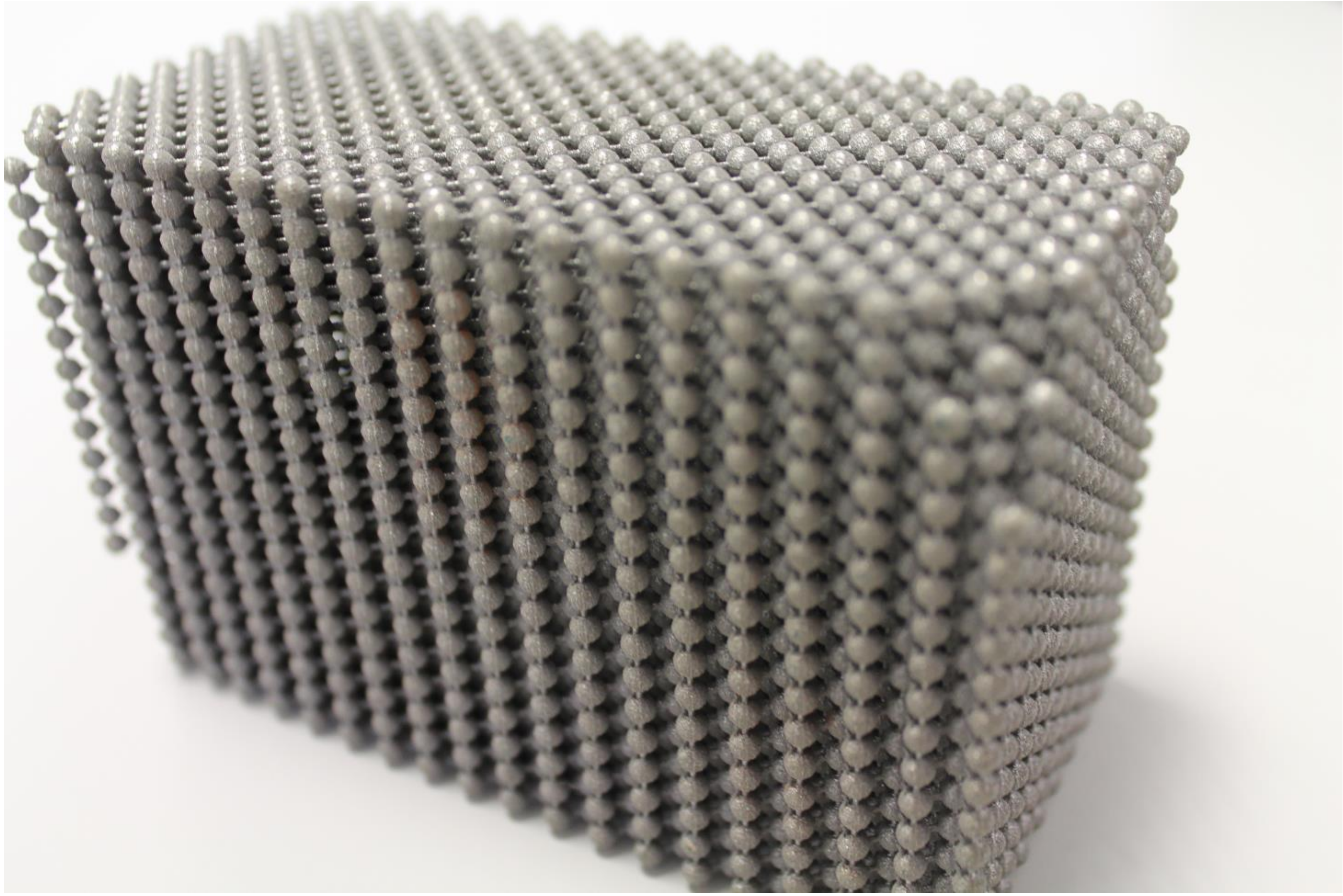








3D nonlinear sonic crystals - experiments



Conclusion

The interplay between nonlinearity and periodicity offers new and interesting possibilities to control wave propagation in structures materials

1D systems allow for analytical predictions based on very simple models

Experiments must be done to confirm the predictions