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Nonlinear acoustic waves in periodic media

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Summary
A high amplitude acoustic wave travelling in a fluid develops higher harmonics as it propagates through the medium. In a homogeneous fluid, such nonlinearly generated harmonics grow during propagation because of the lack of dispersion. On the other hand, periodic acoustic media such as 1D layered media, or 2D sonic crystals, are known to introduce strong dispersion in wave propagation, even creating forbidden propagation bands or bandgaps where wave propagation at particular frequencies is not allowed. The combined action of nonlinearity (harmonic generation) and periodicity (different propagation velocities and attenuation for the different harmonics) results in novel and unexpected phenomena with respect to the linear counterpart, and opens the door to new mechanisms of acoustic wave control and manipulation. Here we investigate numerically the propagation of a plane acoustic wave in a periodic medium in two situations: a structured fluid, formed by a periodic array of fluid layers with alternating acoustic properties, and a 2D squared array of solid scatterers embedded in a fluid (a sonic crystal). We show how the nonlinear generation and propagation of the second harmonic is strongly affected by the presence of the crystal.

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1. Introduction

Sound wave propagation in periodic media has become very popular in the last 20 years in acoustics, after the introduction of the concept of sonic crystals. Exploiting the analogies with other type of waves in the corresponding media (mainly electrons and light) many interesting effects as forbidden propagation bands (band-gaps), focalization, self-collimation, negative refraction, and many others have been proposed. The huge majority of the studies where conductoed for the low-amplitude (linear) regime, where the mentioned analogies apply. Intense wave propagation in nonlinear periodic media has been much less explored. Here we consider different examples of new phenomena related to sound wave propagation in nonlinear medium with periodically modulated properties. We start from the simpler 1D case, or nonlinear multilayered structure, also called a superlattice. Nonlinear acoustical effects in such structure have been studied only in a few works. For example, in [1] the harmonic generation process is described in a fluid/fluid multilayered structure (water/glicerine), based in a nonlinear wave equation. Also, acoustic solitons in solid layered nonlinear media have been presented in [2]. More recently, nonlinearity plus periodicity has been considered in [3], where an asymmetric propagation device (acoustic diode) was proposed. There, the nonlinearity and the periodicity appear in different position and its effect is consider separately. In this work we show different nonlinear effects that are observed as a result of the interplay between nonlinearity and periodicity of acoustic waves. In section 2 the case of 1D periodic systems is considered. Some basic effects as harmonic generation, modulational instability and soliton formation are reported. In Section 3 the case of 2D sonic crystals is considered.

2. One-dimensional case: superlattices

First we investigate nonlinear sound wave propagation in a one-dimensional structure, consisting of periodically arranged (along the propagation axis) layers of elastic isotropic materials of two types, with thickness d_α ($\alpha = 1, 2$), being c_α the velocity of the sound wave in the α layer. The structure is periodic with period $d = d_1 + d_2$. Such a periodic structure is often called an elastic superlattice (SL) [4], since

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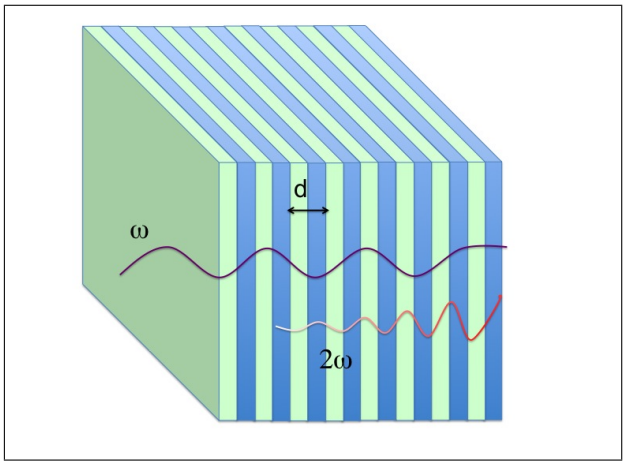


Figure 1. Multilayered structure with period d. The incoming fundamental wave of frequency ω and its nonlinearly generated second harmonic at 2ω are shown for ilustration

it has a macroscopic period d which, by definition, greatly exceeds the interatomic distance. Sometimes a macroscopic periodic structure consisting of alternating elastic materials that differ in their elastic moduli and sound speeds is called a phononic crystal, but this term is usually reserved for 2D and 3D periodic systems.

Phononic crystals are known present strong dispersion and generate spectral regions named phononic band gaps (PBGs) where sound cannot propagate, in a manner analogous to the formation of electronic band gaps in semiconductors or photonic bandgaps for light.

2.1. The model(s)

There are several models to describe acoustic nonlinear waves, see for example the classical books [5, 6]. They are based on wave equations or on constitutive equations. Several nonlinear wave equations have been proposed, with validity restricted to particular conditions. For example, for an ideal gas the particle displacements ξ obey, in lagrangian coordinates a, the equation

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{c_0^2}{(1+\frac{\partial \xi}{\partial a})^{\gamma+1}} \frac{\partial^2 \xi}{\partial a^2}, \tag{1}$$

where γ is the ratio of specific heats. The equation can be generalized to any fluid with nonlinearity B/A just replacing $\gamma = 1 + B/A$. For small displacements, it is also possible to expand the nonlinear operator to get an approximate equation at the quadratic order.

If the wave propagates in an inhomogeneous medium, the parameters c_0 and B/A can be treated as space-dependent (periodically distributed in the case of a crystal). This equation was solved in [1] to study harmonic generation in the multilayer.

In a different approach, less restrictive, the propagation of sound can described by the Navier-Stokes equations, namely the continuity and Euler equations, completed by the equation of state, which in 1D read

$$\begin{aligned} &\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v)}{\partial x} \\ &\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = -\frac{\partial P}{\partial x} \\ &P = P(\rho) \end{aligned}$$

where $P = p_0 + p'$ is the pressure, v is the particle velocity (in general a vector, here a scalar), $\rho = \rho_0 + \rho'$ is the density. Note that p is called the acoustic pressure, and subscript 0 refers to equilibrium values.

Apart from the convective nonlinearity in the Euler equation, also an acoustical nonlinearity appears in the state equation. Expanding up to second order, we deal with a quadratic nonlinearity which is dominant for most of the fluids

$$p = c_0^2 \rho' + \frac{c_0^2}{\rho_0}\frac{B}{2A}\rho'^2 \tag{2}$$

where, c_0 the (linear) sound speed, and B/A is the nonlinearity parameter. Again, the medium parameters ρ_0, c_0 and B/A can be considered as space-dependent (periodic in the case of a crystal).

2.2. (Linear) Dispersion Relation for a superlattice

Dispersion relation, or band structure, for 1D systems can be expressed analytically. In 2D, only numerical solutions (Plane wave expansions,...) or aproximate (valid under particular conditions) are possible. It is given by the so called Rytov formula [4],

$$\begin{aligned} &\cos(kd) = \cos(k_1 d_1)\cos(k_2 d_2) - \\ &\frac{1}{2}\left(\frac{k_1}{k_2} - \frac{k_2}{k_1}\right)\sin(k_1 d_1)\sin(k_2 d_2) \end{aligned} \tag{3}$$

where k is the Bloch wave-number, and $k_j = \omega/c_j$. For a given input frequency ω and a for a medium with known acoustical c_j and geometrical d_j parameters, Eq. (3) results in a band structure of propagating and nonpropagating (bandgap) regions as shown in Fig.(1).

2.3. Harmonic generation

The most basic nonlinear effect is harmonic generation. This problem has been treated in [1], we present here a more accurate study based on solutions of the constitutive equations, and report some new results.

We consider a fundamental wave at a frequency slightly below the half of the first band. It nonlinearly generated second harmonic lies then close to

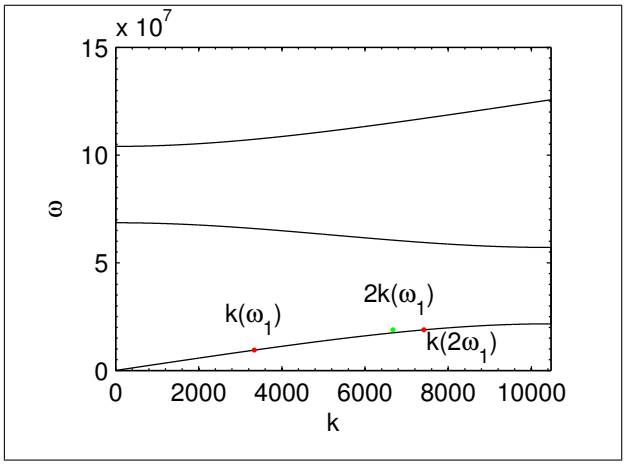


Figure 2. Band diagram, showing the first three propagation bands, and the wavenumbers involved in the second harmonic generation process. The parameters are the thickness of the layers $d_1 = d_2 = 0.1$ mm and sound velocities $c_1 = 1500$m/s and $c_2 = 500$m/s

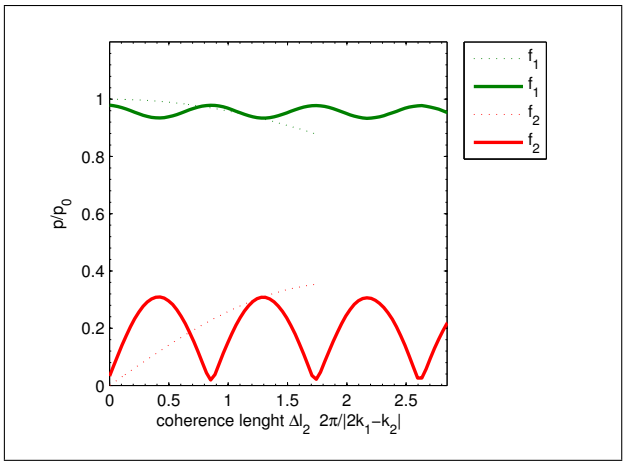


Figure 3. First and second harmonic amplitudes along the propagation distance, normalized to the coherence length. Input amplitude is 50 MPa

the edge of the band-gap, but still in the propagation band. Two second harmonic components with different wavenumbers are present: a forced wave at $k(2\omega)$ and a propagating wave at $2k(\omega)$, that fulfills the dispersion relation. As a result of the interaction, the second harmonic total field experience beatings or low-frequency oscillations. The beating period is given by the mismatch $\Delta k = k(2\omega) - 2k(\omega)$, which defines the so called coherence length, $l_c = 2\pi/|\Delta k|$. Figure shows the propagation of an input wave with frequency $f = 1$ MHz and strong input amplitude $50MPa$, and its second harmonic at 2 MHz in the region of large dispersion (see Fig 2). We observe that at high intensities the beating period becomes amplitude dependent. This means that the beating period is less than the coherence length, as occurs at low amplitudes. It is a signature of nonlinearity of dispersion. Figure 4 shows the amplitude of second harmonics in

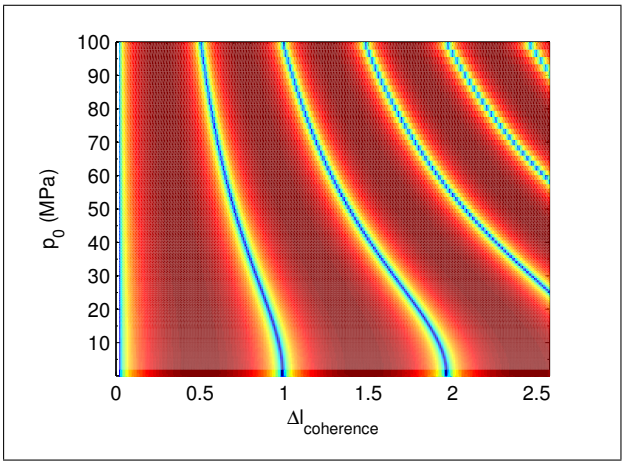


Figure 4. The amplitude of second harmonics along the propagation distance, as the input amplitude is increased.

space, as the input amplitude is increased. The decrease of the beating period is evident.

2.4. Modulational instability and solitons

Other typical nonlinear effect of dispersive systems is the formation of solitons. They are expected when dispersion is compensated by nonlinearity. In the weakly dispersive regime (long wavelength limit), dispersion can be well represented by a linear plus a cubic term, in the form $\omega = ck + \beta k^3$, which allows to derive in the continuous limit a Korteweg de Vries (KdV) equation, which is known to support solitons in the form $sech(x)$. Numerics confirm the existence of solitons in this regime, as shown in Fig. 5.

3. Two-dimensional case: sonic crystals

When we deal with two dimensional systems, or sonic crystals, additional degrees of freedom appear: the symmetry of the lattice, the finite size and transverse structure of the beams, etc. A squared lattice symmetry is considered here for simplification. Note also that, unlike the one dimensional case, in 2D the scatterers can be considered rigid, which further simplifies the problem. Sound propagation in the material is ruled by the following constitutive equations , which are an extension of the 1D system of the preceding section

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho v), \tag{4}$$

$$\rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p \tag{5}$$

where $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and we have ignored dissipative effects.

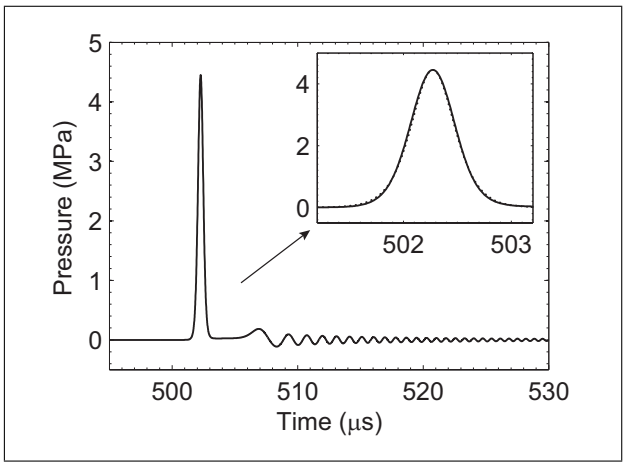


Figure 5. Acoustic soliton. Input is a gaussian pulse, with central frecuency 1 MHz and amplitude 6 MPa. Thickness of each layer is $d = 0.1/\lambda = 0.15$ mm. Layers have the same density and nonlinearity, $\rho = 1000Kg/m^3$ and $B/A = 5$, but different sound velocities $c_1 = 1500$m/s and $c_2 = 500$m/s. $N = 2000$ layers were considered (material length is $L = 30$ cm)

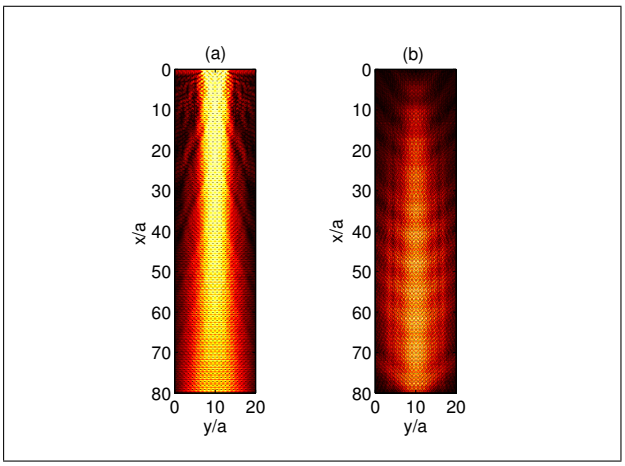


Figure 6. Self-collimation of fundamental and second harmonics

Equations (5) must be closed, as in the 1D case, with the quadratic nonlinear relation, Eq. (2). Similarly, the medium parameters ρ_0, c_0 and B/A can be considered as space-dependent in (x, y) (periodic in the case of a crystal).

3.1. Self-collimation of nonlinear beams

One of the most striking and discussed effects in wave propagation in periodic media has been called self-collimation. At a particular frequency, a beam can propagate through the crystals without apparent diffraction [7]. Simultaneous self-collimation of (linear) first and second harmonics was demonstrated in [8]. However, and intense beam will not collimate in general unless the second harmonic fulfill a number of conditions as (1) flatness of the isofrequency contours for second harmonics, (2) phase matching between first and second harmonics, and (3) nonnegligible spatial coupling between harmonics. Numerical simulations of the nonlinear equations show that first and nonlinearly generated second harmonics can in fact propagate collimated.

4. Conclusion

We have intestigated theoretically the propagation of sound waves in a nonlinear periodic medium. We ilustrate some examples of new phenomena in materials with one- and two-dimensional modulation of the acoustical parameters. For one-dimensional systems, the so called superlattices or multilayers, we report the nonlinear generation of second harmonics, and how its propagation is affected by dispersion. In particular, we observe a nonlinear dependence of the dispersion relation. We report also the existence of solitons of KdV type in this system. In two-dimensional structures, we describe the process of simultaneous beam self-collimation of the fundamental wave and its non-linearly generated second harmonics, and describe the conditions for its realization.

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References

[1] Y. Yun, G.Q. Miao, P. Zhang, K. Huang, R.J. Wei, Nonlinear acoustic wave propagating in one-dimensional layered system, Phys. Lett. A. 343, 351-358 (2005)

[2] R.J. Leveque and D.H. Yong, Solitary waves in layered nonlinear media, SIAM J. Appl. Math. 63, 1539âĂŞ1560 (2003)

[3] B. Liang, B. Yuan, and J.-ch. Cheng, Acoustic Diode: Rectification of Acoustic Energy Flux in One-Dimensional Systems, Phys. Rev. Lett. 103, 104301 (2009)

[4] A.M. Kosevich, *The Crystal Lattice: Phonons, Solitons, Dislocations, Superlattices*, Second Edition. WILEY-VCH Verlag (2005), Chapter 5

[5] M.F. Hamilton and D.T. Blackstock, *Nonlinear Acoustics*, Academic Press (1998)

[6] K. Naugolnykh and L.A. Ostrovskii, *Nonlinear wave processes in acoustics*, Cambridge University Press (1998)

[7] I. Perez-Arjona, V. J. Sanchez-Morcillo, J. Redondo, V. Espinosa, and K. Staliunas, Phys. Rev. B 75, 014304 (2007).

[8] E. Soliveres, I. Perez-Arjona, R. Pico, V. Espinosa, V. J. Sanchez-Morcillo and K. Staliunas, Simultaneous self-collimation of fundamental and second-harmonic in sonic crystals, Appl. Phys. Lett. 99, 151905 (2011)