TOY MODELS FOR NONLINEAR ACOUSTICS

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Nature, models and toys

Nature: complex dynamical systems

Toy model: a simple model for a complex system

- Useful when complex experiments and/or simulations are needed
- Math simplification
- Analogies help to understand the complex system
- In some cases you can build the toy model and develop experiments

Two cases:

(1) The mechanical analogy of a microbubble under ultrasound field

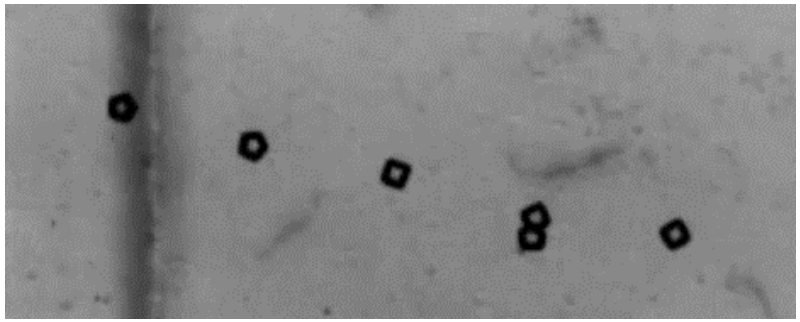
(2) A magnetically coupled pendula chain as a model for nonlinear waves

TOY MODEL 1

a ring of couple pendula
as analogy of
microbubbles under ultrasound field

Microbubbles under ultrasound

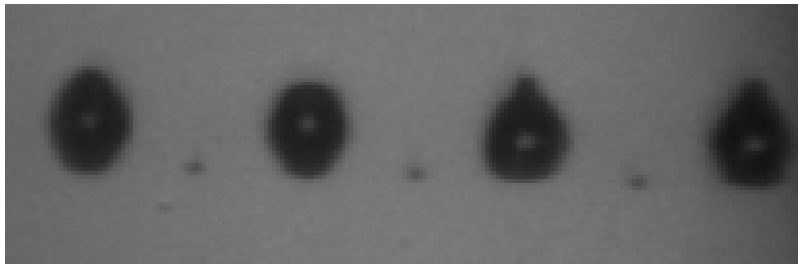
- Gas bubbles driven by an ultrasound field develop surface modes.



- Mechanism: parametric instability



- There are also evidences of localized modes in bubbles and non-symmetric oscillations



The analogy

Gas bubble: Radial mode dynamics by Rayleigh-Plesset eq.

$$\rho\left(R\ddot{R}+\frac{3}{2}\dot{R}^2\right)=p_g-p_0-p_a(t)-\frac{2\sigma}{R}-4\mu\frac{\dot{R}}{R} \qquad p_a(t)=p_A\cos(\omega t)$$

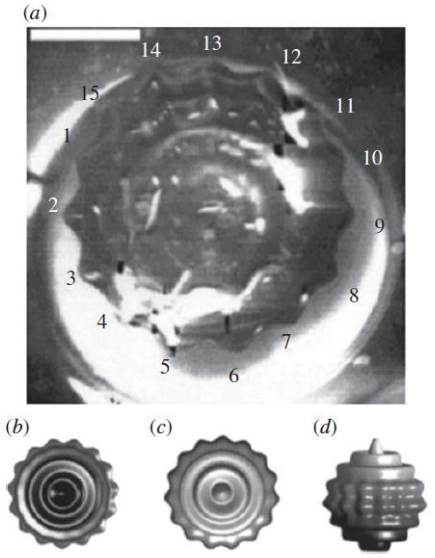
- Spherical modes projection

$$R(\theta,t)=R(t)+\sum_n a_n(t)Y_n(\theta)$$

$$\ddot{a}_n+2\gamma\dot{a}_n+\omega_n^2 a_n=0$$

$$\gamma=\frac{3\dot{R}}{R}+\frac{2(n+2)(2n+1)\mu}{\rho R^2}$$

$$\omega_n^2=\frac{(n-1)(n+1)(n+2)\sigma}{\rho R^3}+\frac{2(n-1)(n-2)\mu\dot{R}}{\rho R^3}$$



J. Acoust. Soc. Am., Vol. 131, No. 6, June 2012

The analogy

- Spherical modes projection, variable change

$$b(t) = a(t)R^{3/2}$$

- Mathieu equation for each surface mode

$$\ddot{b}_n + \left[\omega_n^2 + \left(\frac{(2n+1)\omega_e^2}{2} - 3\omega_n^2\right)\frac{R_\varepsilon}{R_0}\cos(\omega_e t)\right] b_n = 0$$

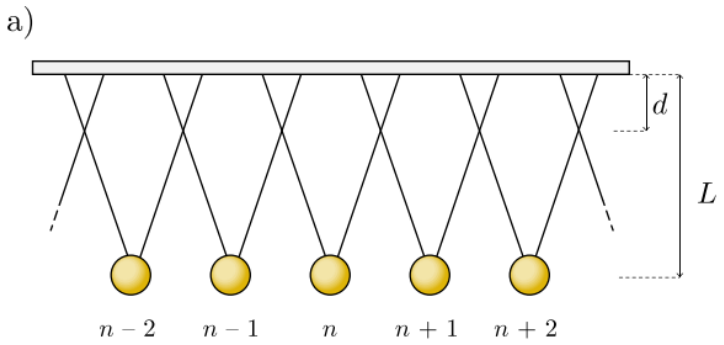
- Dispersion relation

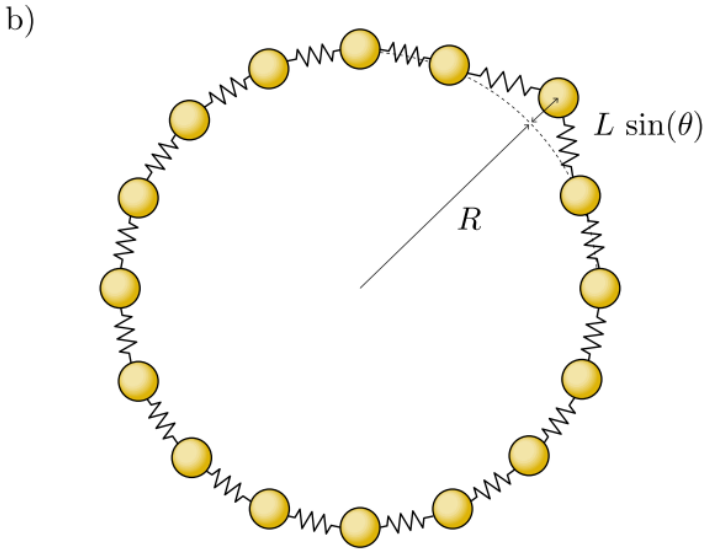
$$\omega_n^2 = \frac{(n-1)(n+1)(n+2)\sigma}{\rho R_0^3}$$

J. Acoust. Soc. Am., Vol. 131, No. 6, June 2012

The analogy

Pendulum ring:





$$\ddot{\Theta}_n + (\omega_0^2 + \eta \cos \omega_e t) \sin \Theta_n - c^2(\Theta_{n+1} - 2\Theta_n + \Theta_{n-1}) = 0,$$

$$\omega_n^2 = \frac{g}{L-d}\left[1 - \frac{d}{L}\cos^2\left(\frac{na}{2R_0}\right)\right]$$

The analogy

Coupled pendula ring:

$$\ddot{\Theta}_i + (\omega_0^2 + \eta\cos\omega_e t)\sin\Theta_i - c^2(\Theta_{i+1} - 2\Theta_i + \Theta_{i-1}) = 0,$$

$$\omega_n^2 = \frac{g}{L-d}\left[1 - \frac{d}{L}\cos^2\left(\frac{na}{2R_0}\right)\right]$$

- Continuous description

$$\ddot{\Theta} - c^2\Theta_{xx} + (\omega_0^2 + \eta\cos(\omega_e t))\Theta = 0$$

Mathieu equation

$$\ddot{b}_n + \left[\omega_n^2 + \eta\cos(\omega_e t)\right] b_n = 0$$

- Dispersion relation

$$\omega_n^2 = \omega_0^2 + c^2\frac{n^2}{R_0^2}$$

J. Acoust. Soc. Am., Vol. 131, No. 6, June 2012

The analogy

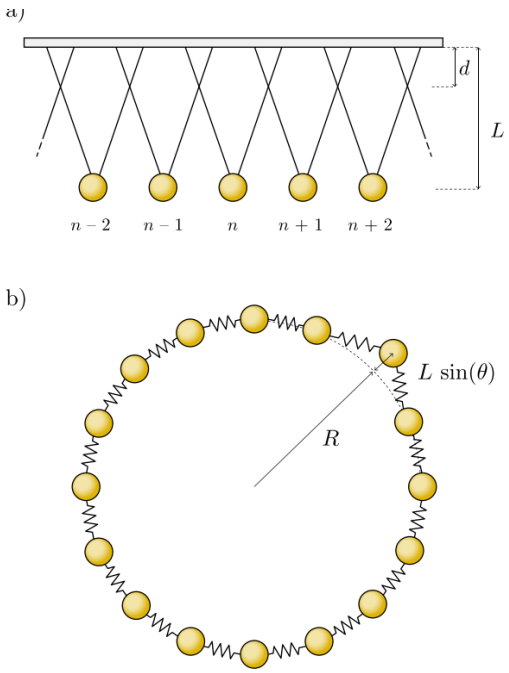
TABLE I. Table of analogies between parameters of microbubble and those of the pendula ring.

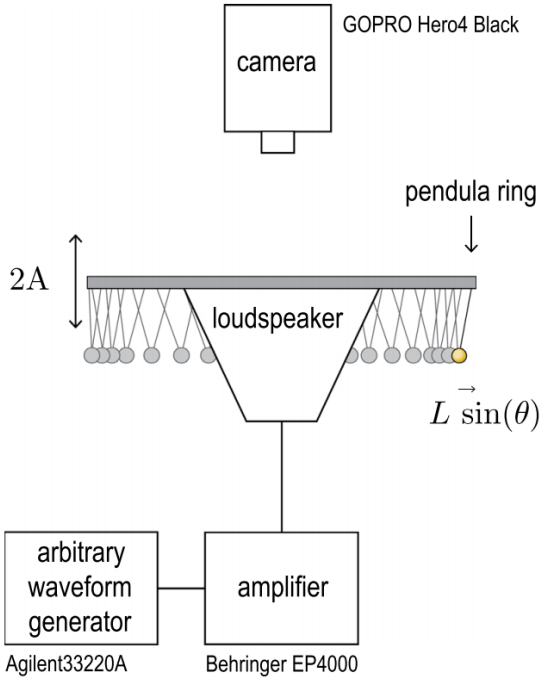
Parameter	Microbubble	Pendula ring
R_0 [m]	$\simeq 10^{-6}$	$\simeq 0.3$
f_e [s^{-1}]	$\simeq 10^{6}$	$\simeq 3$
$f_e \cdot R_0$ [m s^{-1}]	$\mathcal{O}(1)$	$\mathcal{O}(1)$
ω_n^2 [s^{-2}]	$(n-1)(n+1)(n+2)\dfrac{\sigma}{\rho R_0^3}$	$\dfrac{g}{L} + n^2 \dfrac{c}{R_0^2}$
η [s^{-2}]	$\dfrac{(2n+1)}{2}\omega_e^2\left(\dfrac{R_\varepsilon}{R_0}\right)$	$4\omega_e^2\left(\dfrac{h_e}{L}\right)$

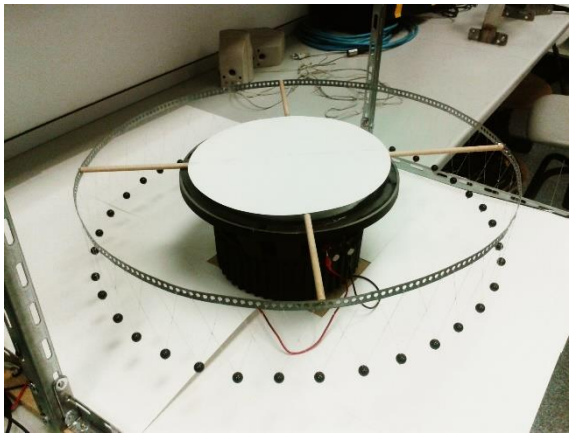
J. Acoust. Soc. Am. 138 (6), December 2015

Ring of coupled pendula

Design and construction

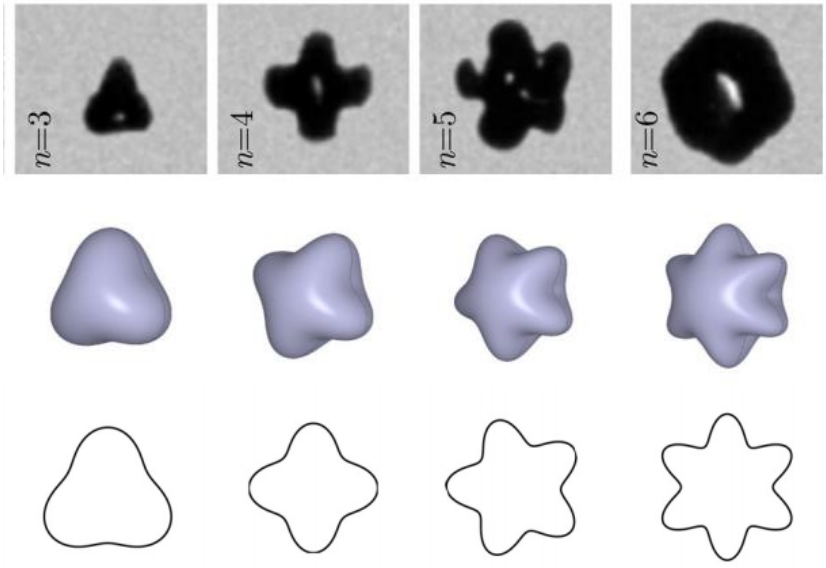


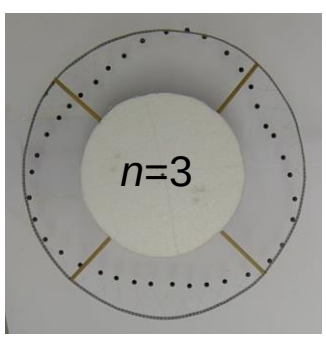




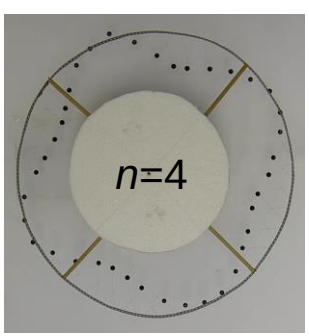
- Acquire displacement waveforms with image processing
- Our chain: N = 50, L = 5 cm, d= 1.6 cm, R = 60 cm

Results (1): Eigenmodes

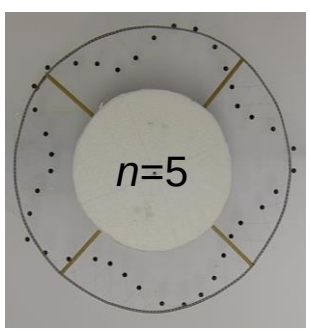




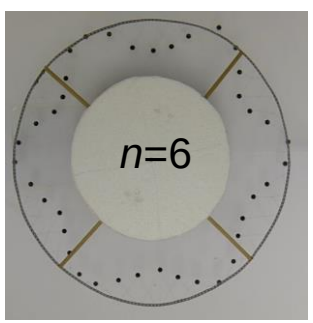
f=3,12 Hz



f=3,15 Hz



f=3,20 Hz

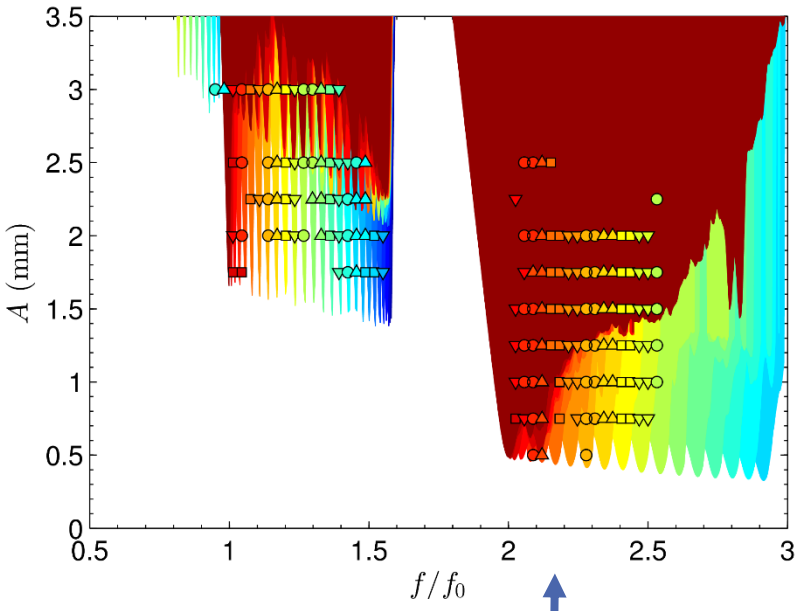


f=3,25 Hz

J. Acoust. Soc. Am. 138 (6), December 2015

Results (1): Mode mixing

- Modes depend on frequency
- Softening nonlinearity
- Signature of nonlinear dispersion



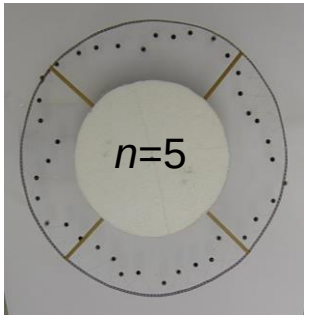
f=3,25 Hz

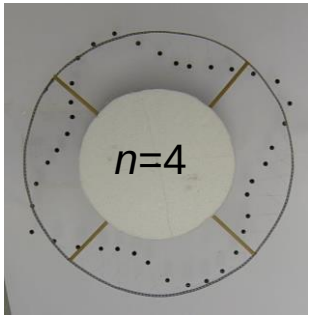
V=500mV

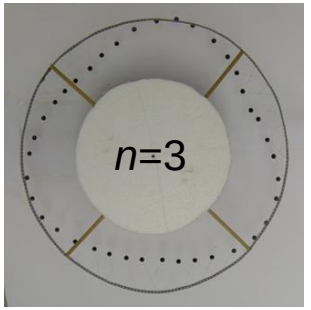
V=250mV

V=80mV

f=3,25 Hz



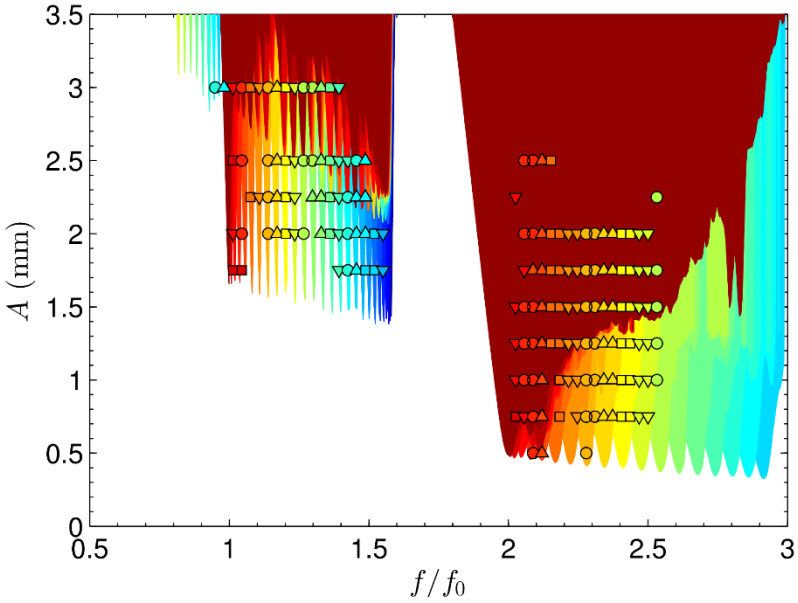


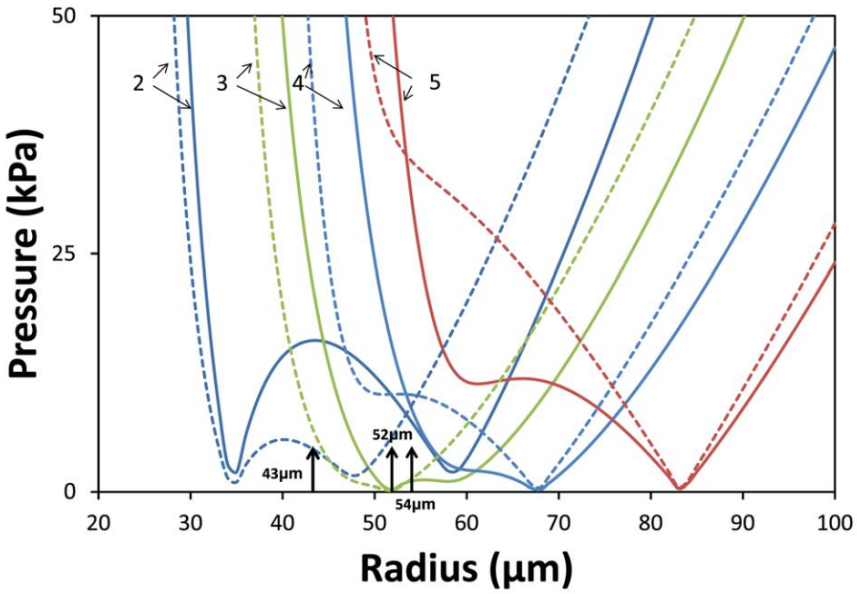


Results (1): Mode mixing

Results (1): modes

- Parametrical excitation: Arnold's tongues for modes



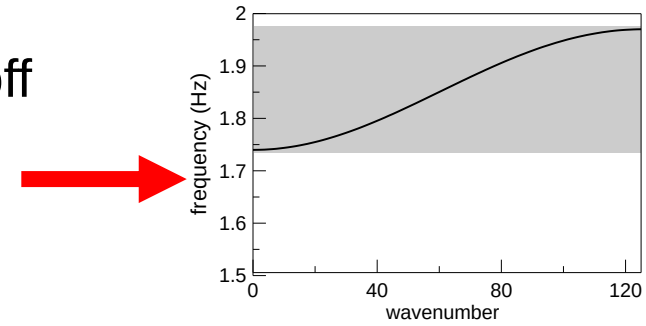


J. Chaline, N. Jimenez, A. Mehrem, A. Bouakaz, S. Dos Santos, V. Sanchez-Morcillo
J. Acoust. Soc. Am. 138 (6), December 2015

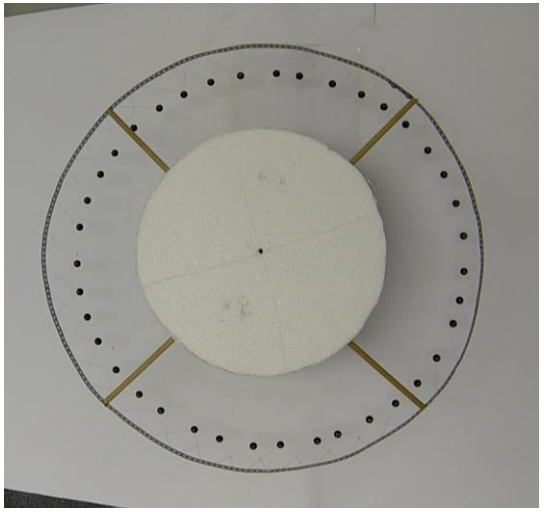
X Xi, F Cegla, R Mettin, F Holsteyns,
J. Acoust. Soc. Am., Vol. 135, No. 4, April 2014

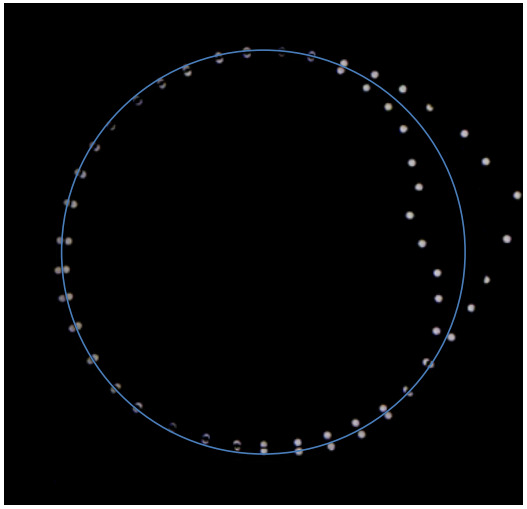
Results (2): Localized modes

- Exist for driving frequencies below lower cutoff frequency



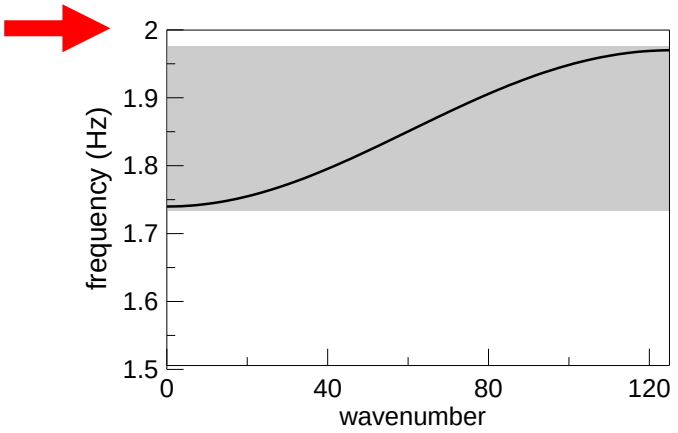
- The shape of the breather in expansion-compression phases is not symmetric



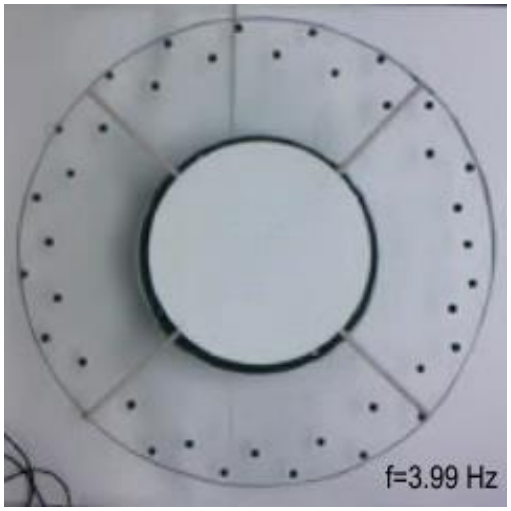


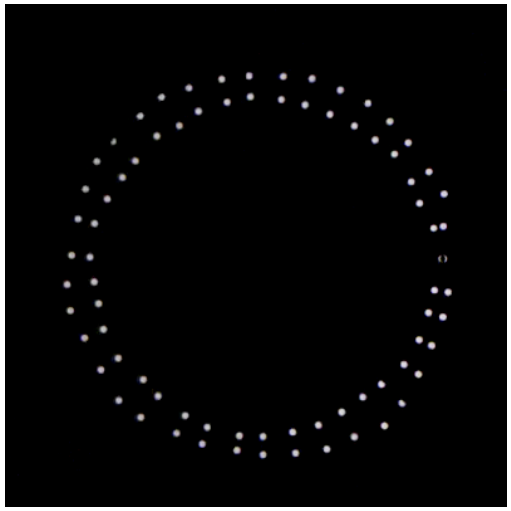
Results (2): Localized modes

- Upper-cutoff kinks



- **Non-physical solution for the bubble!**



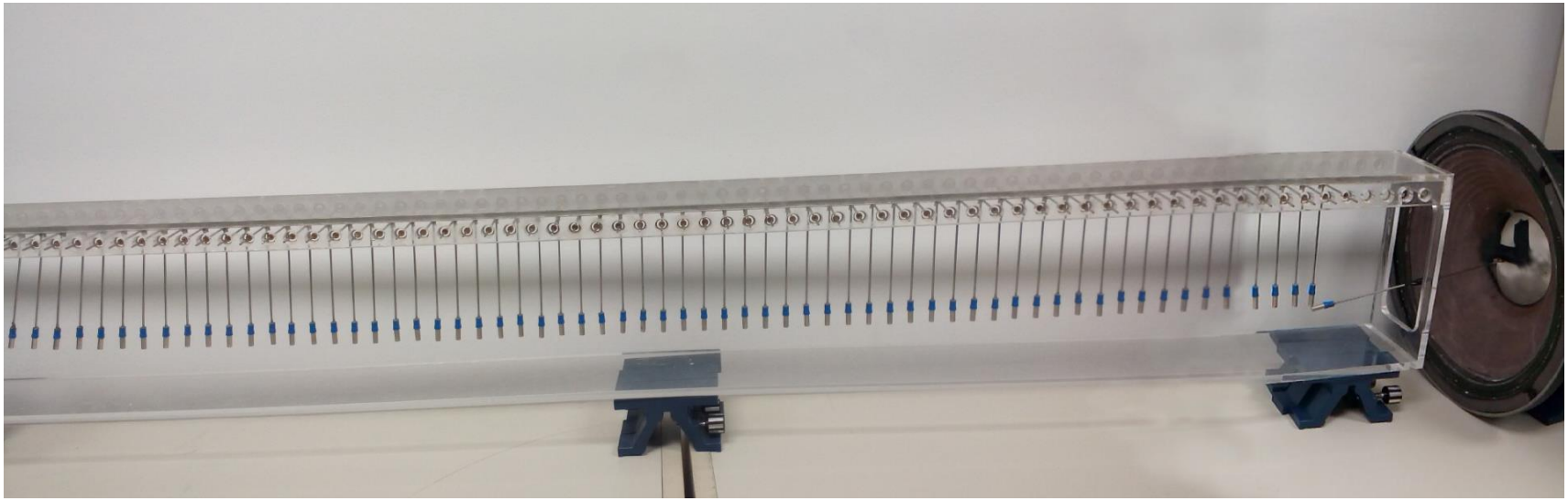


TOY MODEL 2

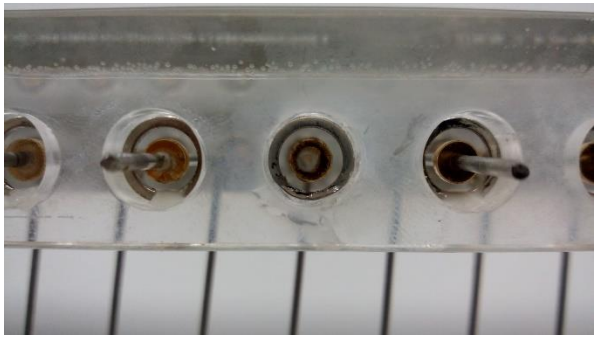
a chain of magnet pendula
as analogy of
lattice waves and multilayer acoustic media

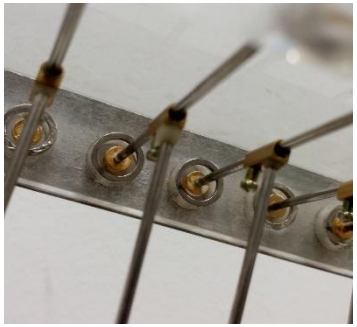
The magnetic chain

Design and construction: N=80 magnets







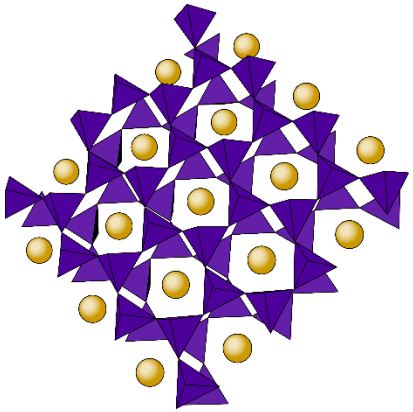


The magnetic chain

- Model of a FPU lattice + restoration (Frenkel-Kontorova)

$$\frac{\partial^2\theta_n}{\partial t^2} - \omega_0^2 \sin(\theta_n) = -c_0^2(\theta_{n+1} - 2\theta_n + \theta_{n-1}) + \alpha\left[(\theta_{n+1} - \theta_n)^2 - (\theta_n - \theta_{n-1})^2\right]$$

- What real systems can be modeled?



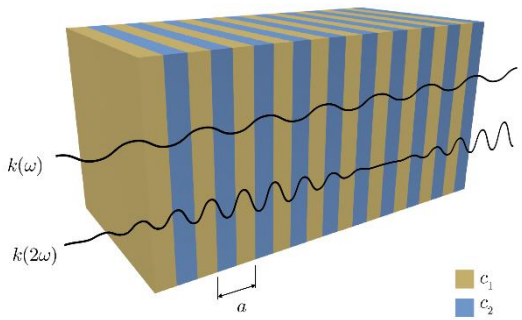
Lattice waves in crystals

Archilla et al.
Physical Review E 91, 022912 (2015)



Granular media

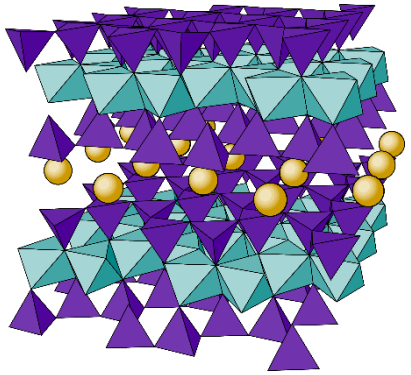
V Sánchez-Morcillo et.al.
Physical Review E 88, 043203 (2013)



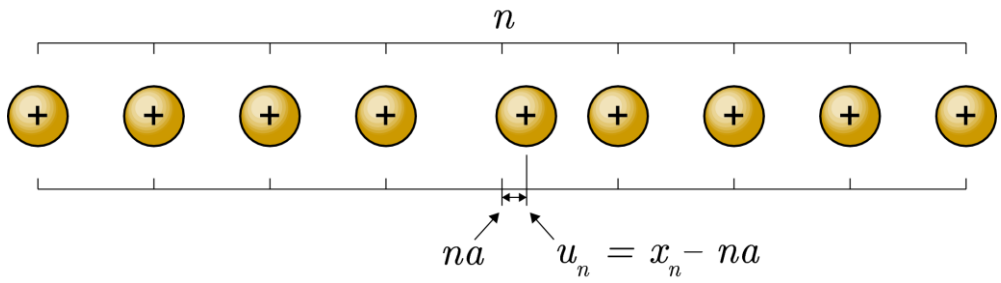
Multilayered acoustic media

N. Jiménez at.al
CRPhys (2016) [accepted In Press]

Lattice waves in crystals



Minimal 1D model for ions in a Coulomb lattice



- Motion equations

$$m_k \frac{\partial^2 x_n}{\partial t^2} = -\frac{Ke^2}{(x_{n+1}-x_n)^2} + \frac{Ke^2}{(x_n - x_{n-1})^2}$$

strain

$$v_n = u_n - u_{n-1}$$

- FPU equivalent

$$\begin{aligned}\frac{\partial^2 u_n}{\partial t^2} = \; & c_0^2[(u_{n+1}+u_{n-1}-2u_n) \\ & -3/2(u_{n+1}-u_n)^2 + 3/2(u_n-u_{n-1})^2 \\ & +2(u_{n+1}-u_n)^3 - 2(u_n-u_{n-1})^3 + \ldots],\end{aligned}$$

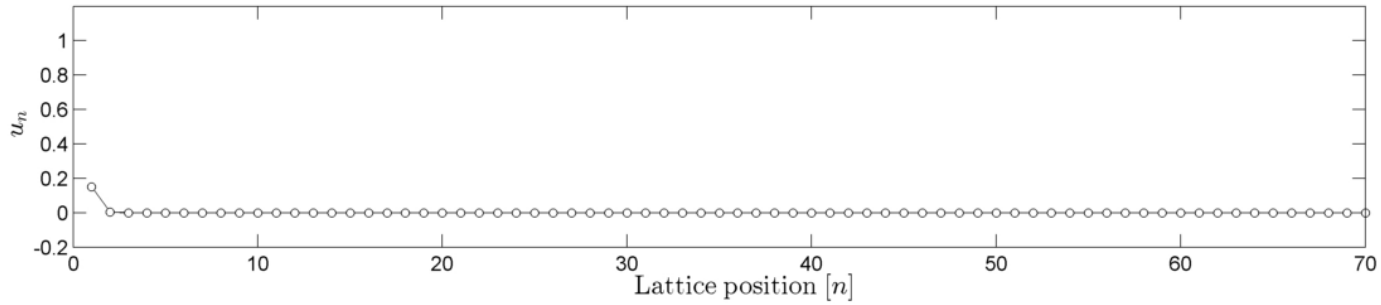
Lattice waves in crystals

Nonlinear localized modes

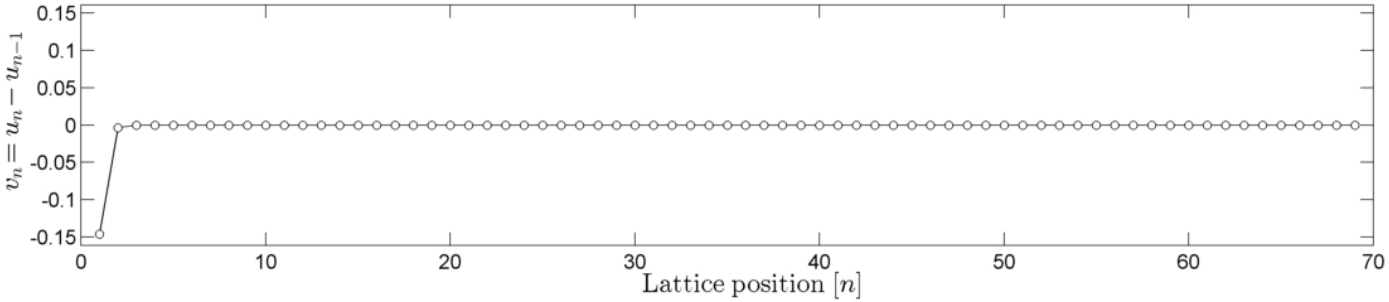
- Small amplitude excitation $u_{n=1}(t) = A_0 \sin(\omega t)$ for $\omega t \leq \pi$



displacement



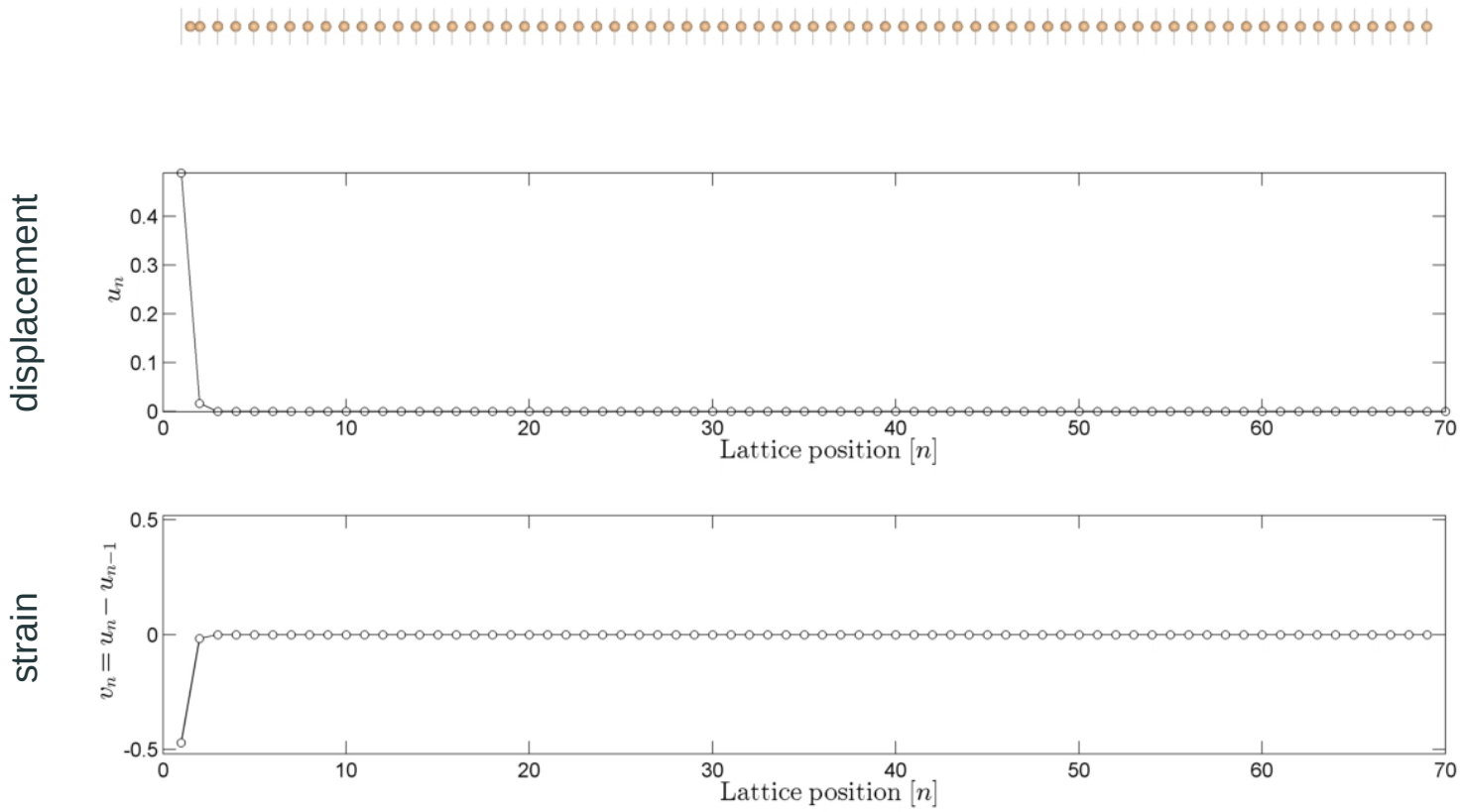
strain



Lattice waves in crystals

Nonlinear localized modes

- Strongly nonlinear regime: ultra-discrete supersonic kink (strain soliton)



Lattice waves in crystals

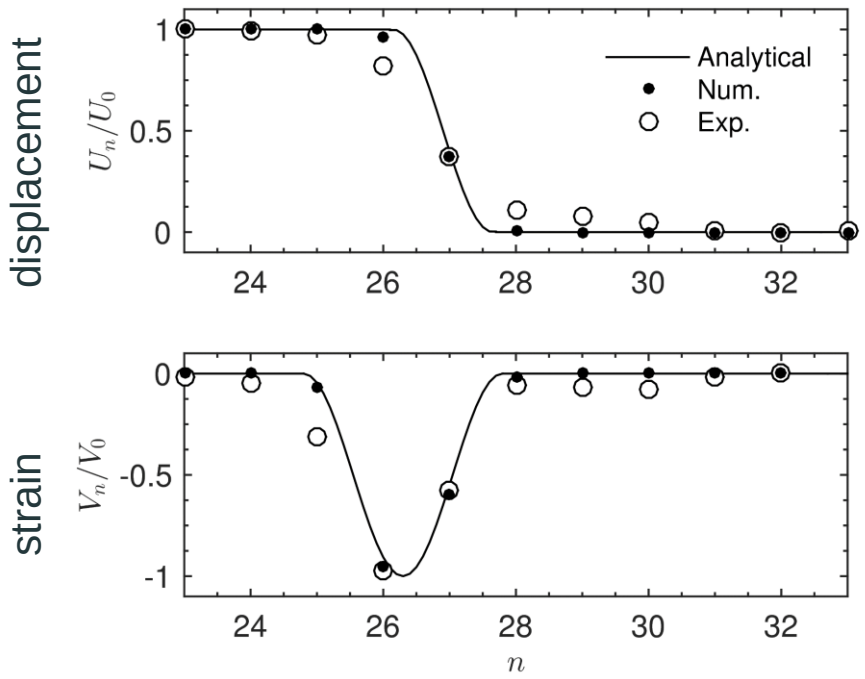
- Experimental setup: example of kink generation

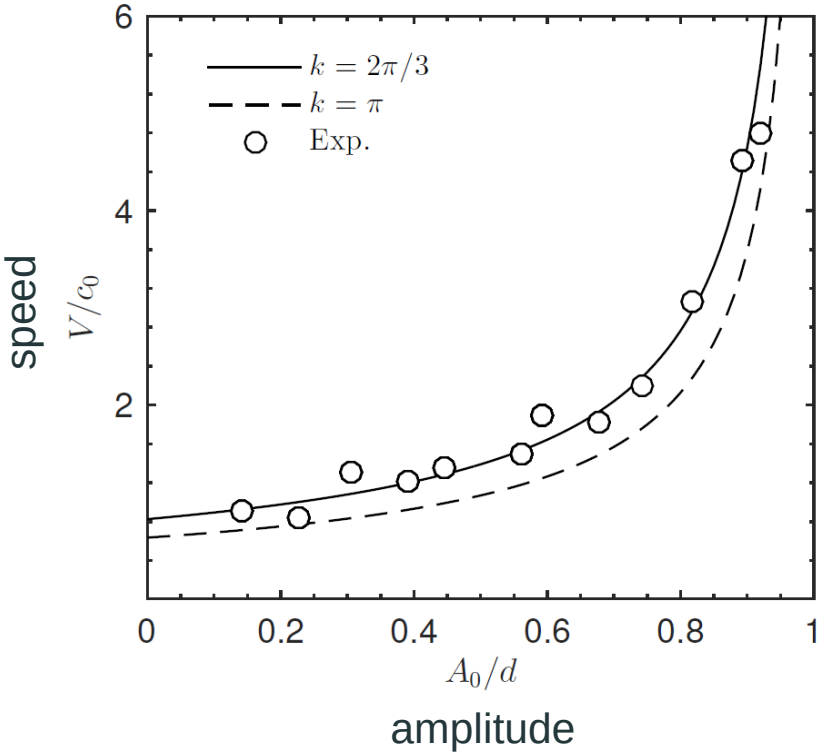




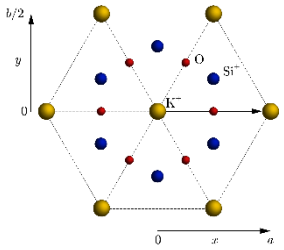
Lattice waves in crystals

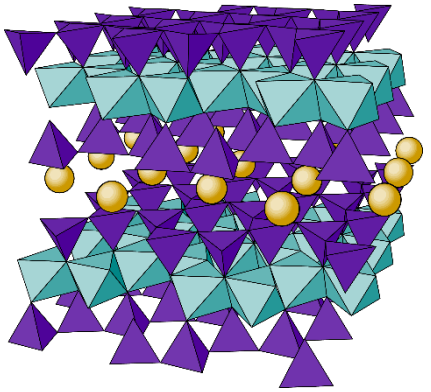
- Amplitude dependent speed is observed
- Supersonic kinks fits the theory for Coulomb-coupled ionic chain





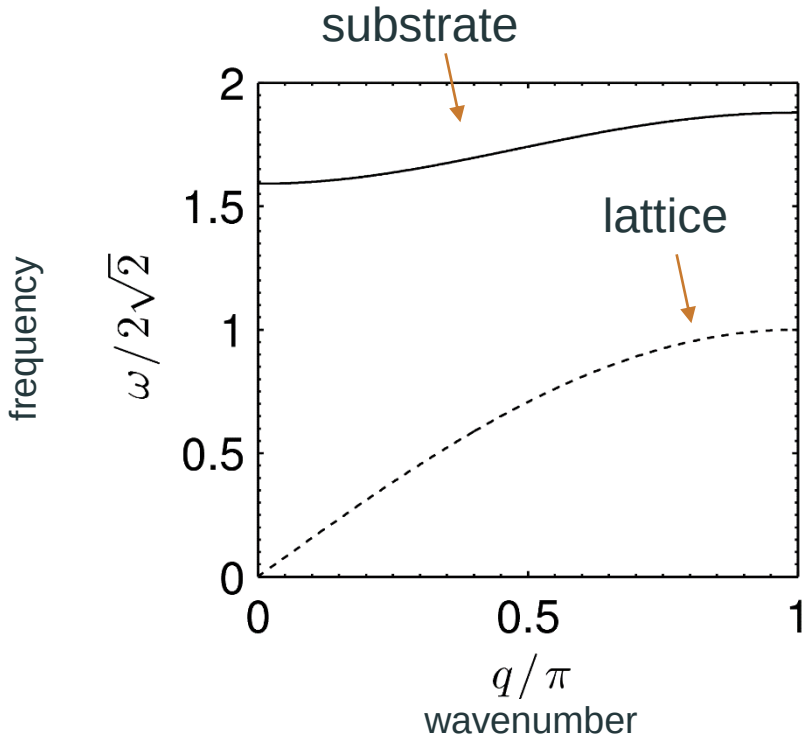
Lattice waves in crystals

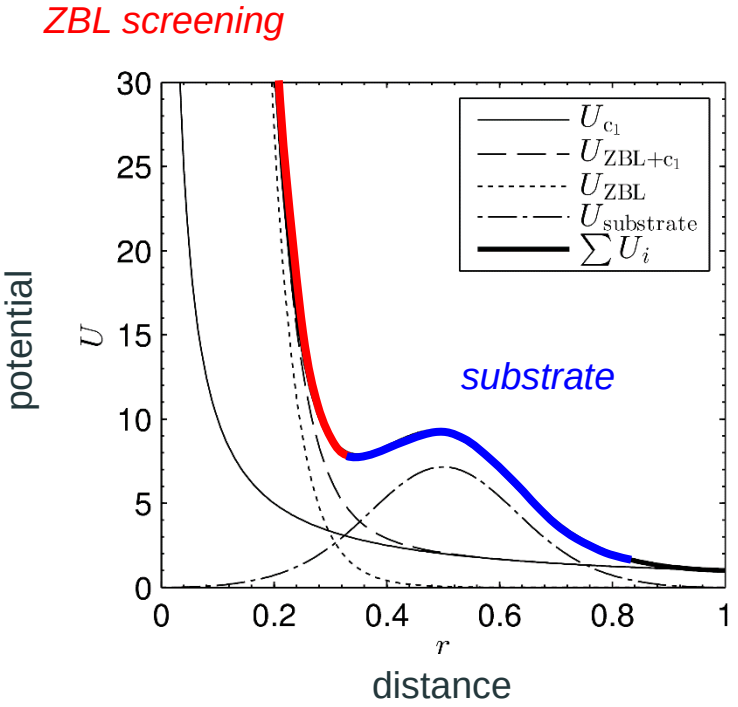




Effect of the crystal: **substrate potential**

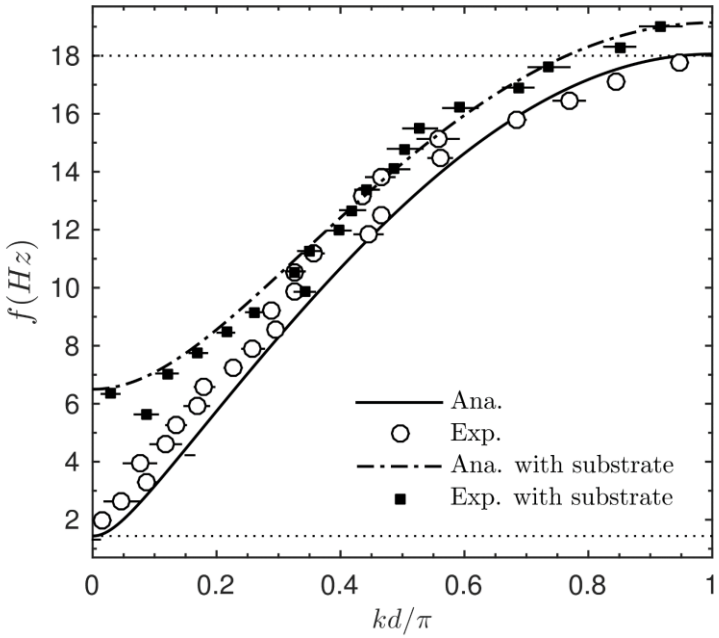
- Muscovite mica mineral
- Assume propagation is 1D: Supersonic excitation

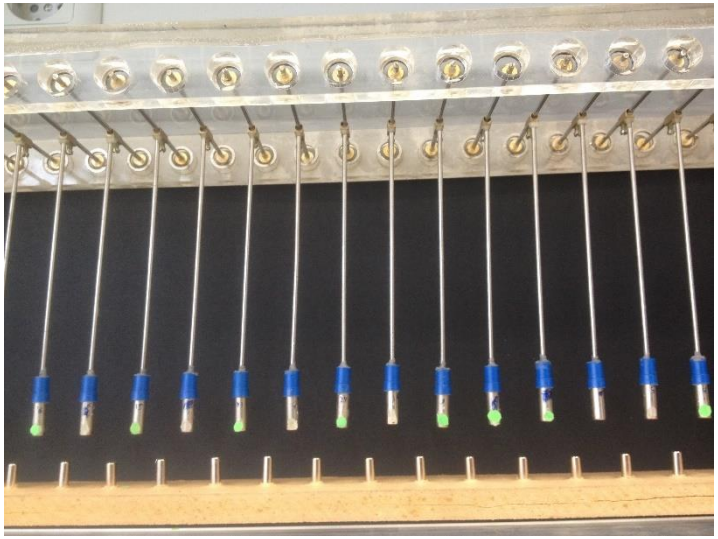




The magnetic chain

Mimic the anharmonic potential: Substrate effect by magnets

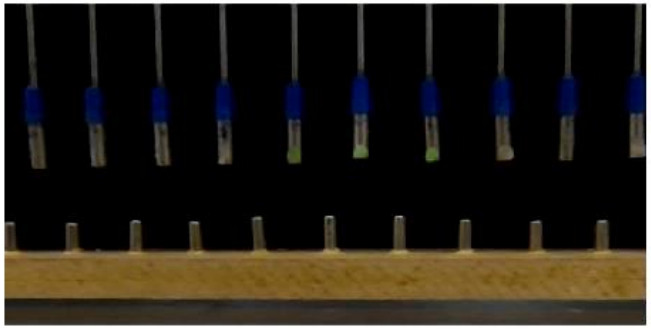


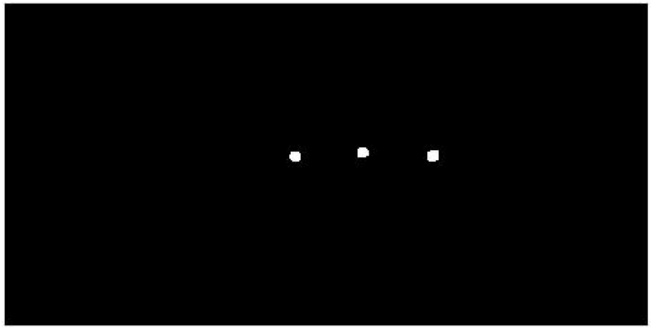


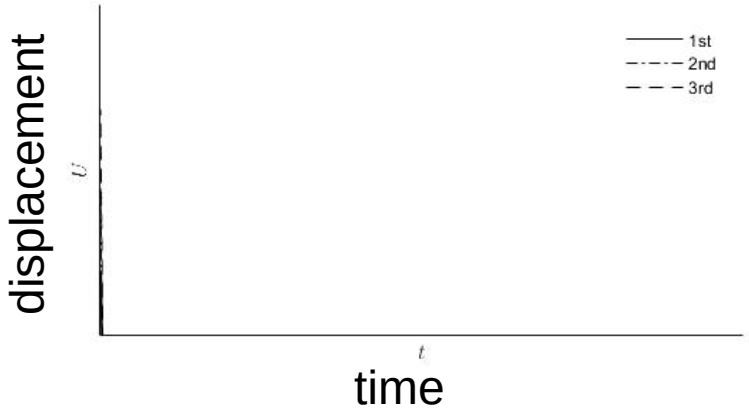
- Fit the crystal dispersion relation and the anharmonic potential

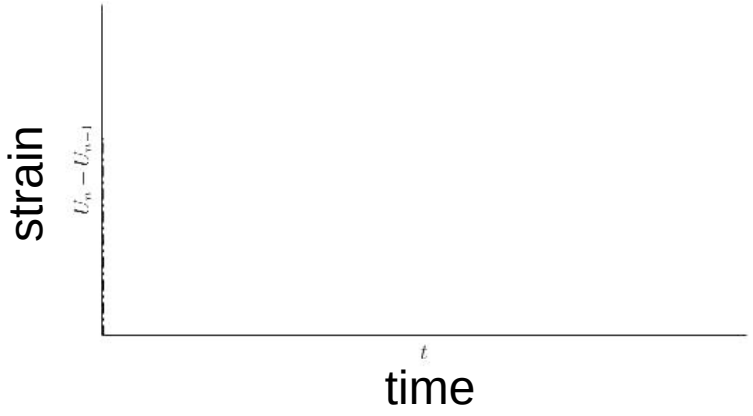
The magnetic chain

- Supersonic kink (crowdion) generation



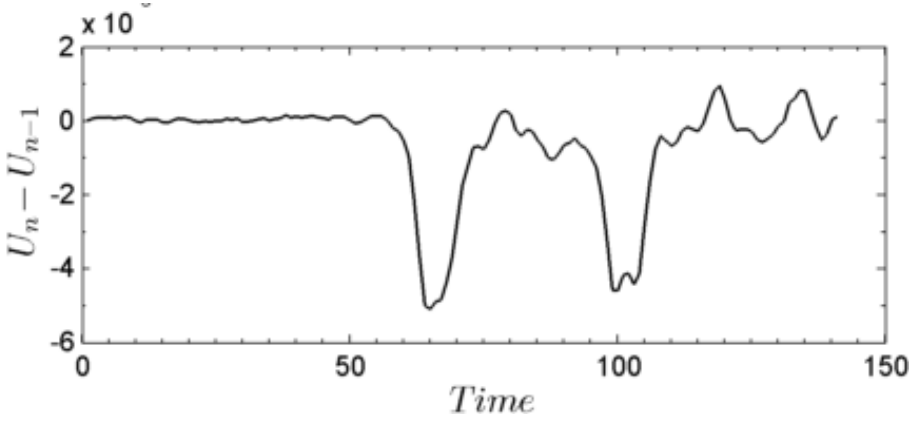


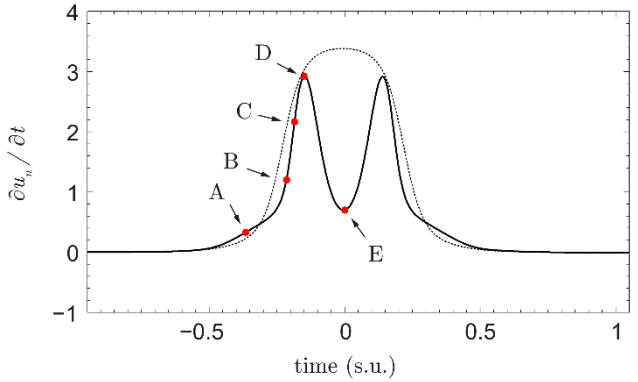




The magnetic chain

- Double strain soliton waveform is achieved in the chain with substrate

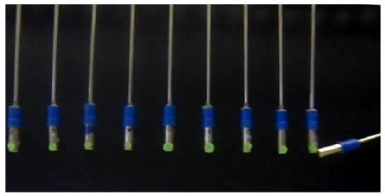


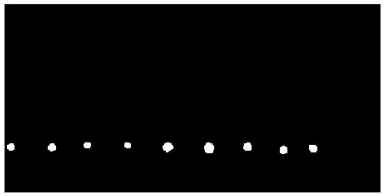


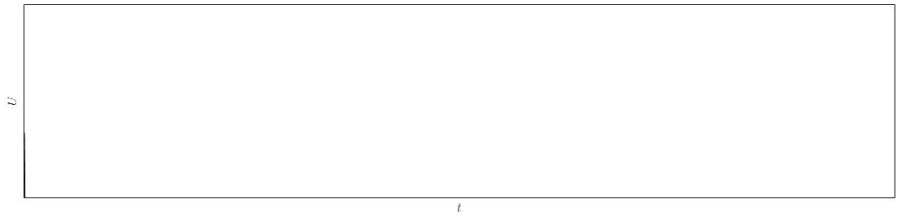
J. F. Archilla, Y. A. Kosevich, N. Jiménez, V. Sánchez-Morcillo, L. M. García-Raffi, *Phys. Rev. E*, **91** (2), pp 022912, (2015)

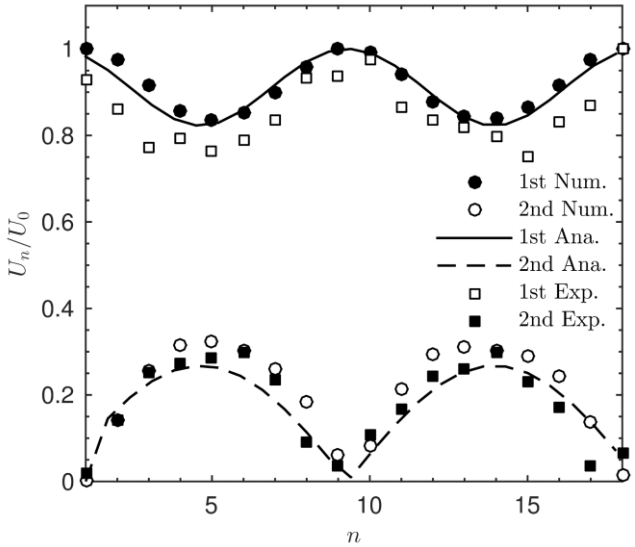
The magnetic chain

Harmonic regime: **nonlinear harmonic generation**





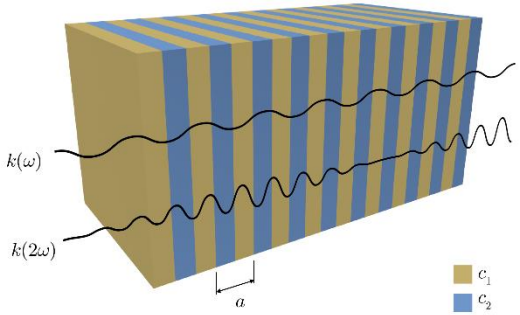






Granular media

V Sánchez-Morcillo et.al.
Physical Review E 88, 043203 (2013)



Multilayered acoustic media

N. Jiménez at.al CRPhys (2016) [accepted In Press]

Conclusions

Toy models are interesting "*per se*"

- A wide range of nonlinear phenomena can be captured
- Explore new features with inexpensive setup

Tools for teaching

- Toy model are a powerful tool to explain basic nonlinear phenomena in a physics lab
- When they work, are even funny!

THANKS FOR YOUR ATTENTION