NONLINEAR WAVES IN PERIODIC MEDIA

FROM LATTICES TO SUPERLATTICES AND PHONONIC CRYSTALS

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V. Tournat, V. Romero García (LAUM, France)

Overview

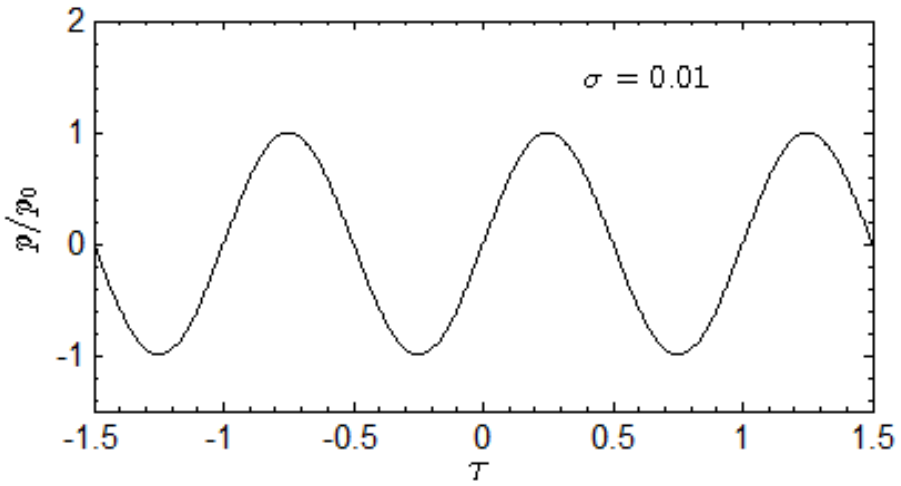
- Introduction:
  - Nonlinear Acoustics: models and phenomena
- Discrete Lattices
  - FPU lattice. Harmonic generation
  - Coulomb lattice: experiments
- Multilayers, Superlattices (1D crystals)
  - Dispersive one-band effects
  - Multi-band effects: Managing of harmonic generation
  - Further control of harmonics: chirped structures
- 2D Sonic crystals
  - Self-collimation

Harmonic distortion in Nonlinear Acoustics

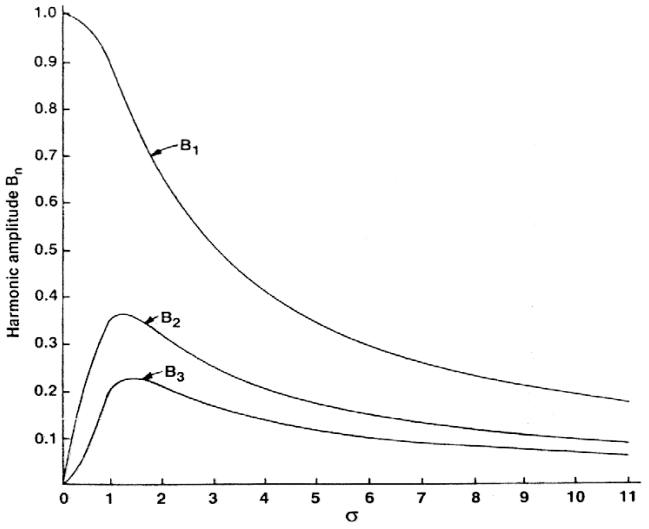
Plane waves in homogeneous medium, nonlinear and nondispersive

$$p_a = p_0 \sum_{n=1}^{+\infty} B_n(\xi) \sin(n\omega\tau)$$

$$B_n(\xi) = \frac{2J_n(n\xi)}{n\xi}$$



Waveform



Spectrum

Equations of Nonlinear Acoustics

Intense sound waves in fluids are accurately described (neglecting dissipation) by the continuity-momentum-state equation system

$$\frac{\partial \rho}{\partial t} = -\frac{\partial (\rho u)}{\partial x},$$

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x},$$

$$p = p(\rho)$$

A quadratic expansion of the state equation leads to

$$p = c_0^2 \rho' + \frac{c_0^2}{\rho_0} \frac{B}{2A} \rho'^2 + \cdots$$

B/A is known as the **coefficient of nonlinearity** of fluids

Equations of Nonlinear acoustics

For 1D plane waves propagating in an ideal gas

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{\left(1+\frac{\partial u}{\partial x}\right)^{\gamma+1}}\frac{\partial^2 u}{\partial x^2}$$

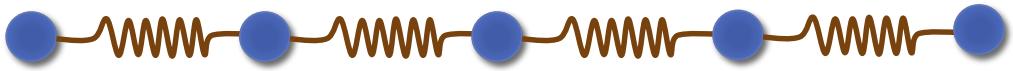
For small displacements

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \beta\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}$$

$\beta = 2 + \frac{B}{A}$ For fluids

$\beta = 3 + \frac{c_{111}}{\rho_0 c^2}$ For solids (along particular directions)

Lattices



The general form (for nearest neighbors coupling)

$$m\frac{\mathrm{d}^2 u_n(t)}{\mathrm{d}t^2} = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1})$$

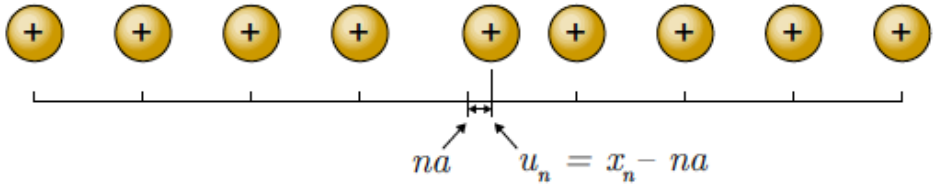
Many possible interaction potentials

$V'(r) = r^{3/2}$ Hertz

$V'(r) = r^{-2}$ Coulomb

$V'(r) = e^{-r} - 1$ Toda

$\vdots$



Most of them approximate to quadratic (FPU) potential

$$V'(r) = kr + br^2$$

FPU Lattices

Consider the lattice with quadratic nonlinearity (α-FPU)

$$\frac{\mathrm{d}^2 u_n(t)}{\mathrm{d}t^2} = k\left(u_{n+1} - 2u_n + u_{n-1}\right) - a\left(u_{n+1} - 2u_n + u_{n+1}\right)\left(u_{n+1} - u_{n-1}\right)$$

For long wavelengths compared with particle spacing, the continuum limit applies, and the evolution equation reads

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = b\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}$$

The nonlinear acoustics wave (Boussinesq) equation!

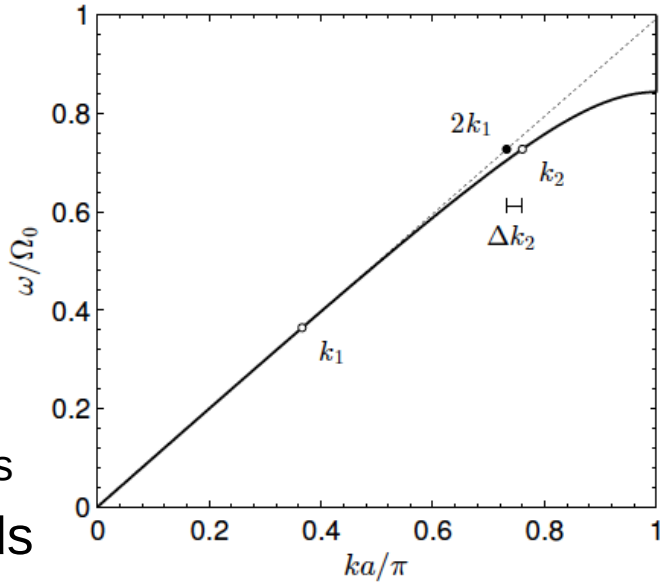
Harmonic generation in Lattices

Main theoretical results for FPU lattice *:

- **Nonlinearity:** Higher harmonic generation (ω, 2ω, 3ω,…)
- **Dispersion:** Two regimes
  - Harmonics in band: beatings at the half period given by the coherence length

$$x = \frac{p}{|Dk|} = \frac{p}{|k(2w) - 2k(w)|}$$

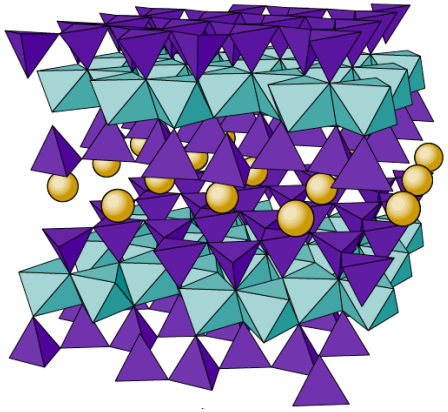
  - Evanescent harmonic: finite amplitude (forced) waves
- **Analytical solutions** by perturbative methods

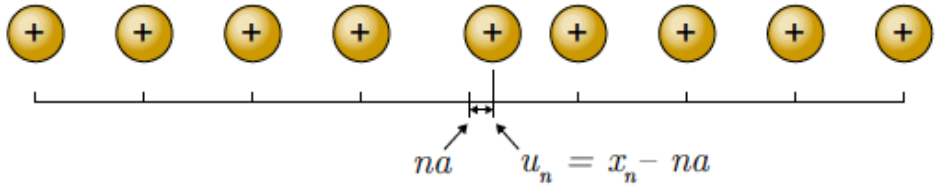


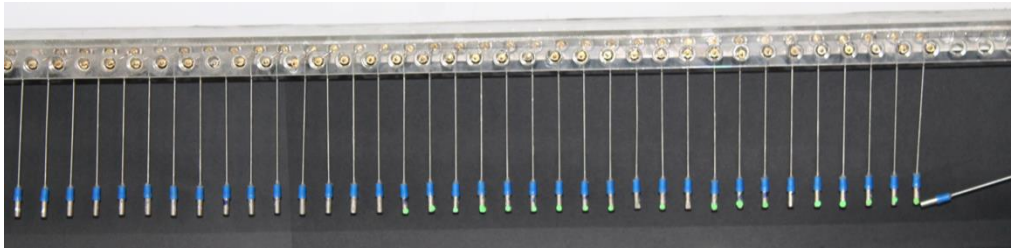
* V. J. Sanchez-Morcillo et al., Second-harmonic generation for dispersive elastic waves in a discrete granular chain, Phys. Rev. E **88**, 043203 (2013)

Coulomb lattice: motivation

Layered silicates
(e.g. Mica Muscovite)*







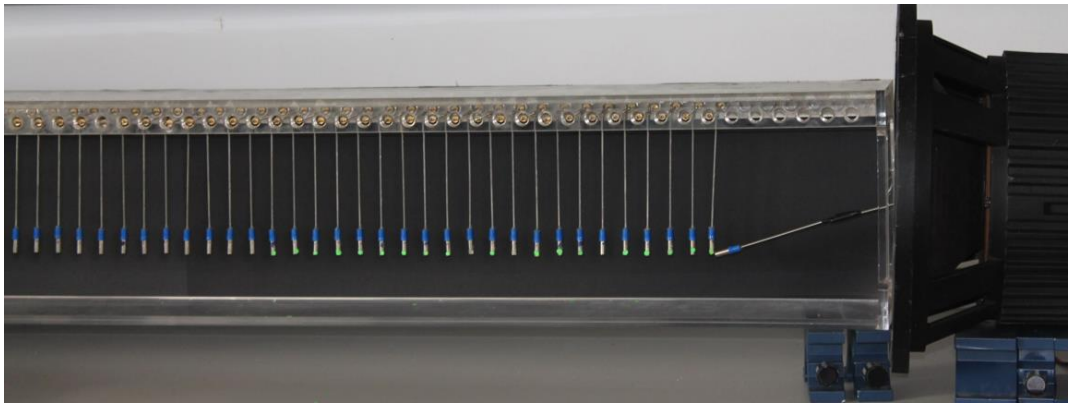
Two motivations:

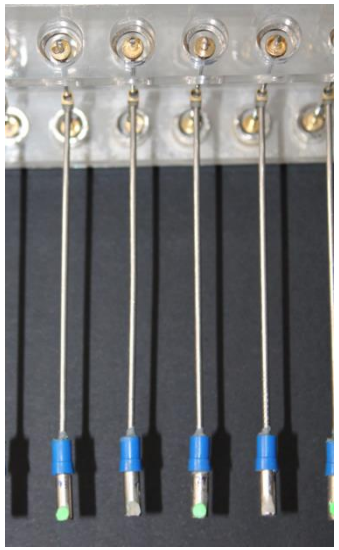
- Macro experiments for the micro world
- Check fundamental results on lattice dynamics

* J.F.R. Archilla, Y. Kosevich, N. Jiménez, V. J. Sanchez-Morcillo, L.M. García Raffi, Ultradiscrete kinks with supersonic speed in a layered crystal with realistic potentials, Phys. Rev. E **91**, 022912 (2015)

Experiment: Coulomb lattice

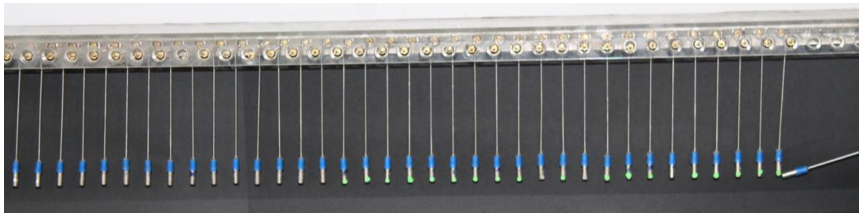
- Array of 50 magnets, with magnetic dipoles vertically oriented in the same direction: repelling forces
- Each magnet is the mass in a pendulum with very low friction
- Pendulums play the role of on-site (substrate) potential.
- Low frequencies: eye view and image processing







Coulomb lattice



Equation of motion

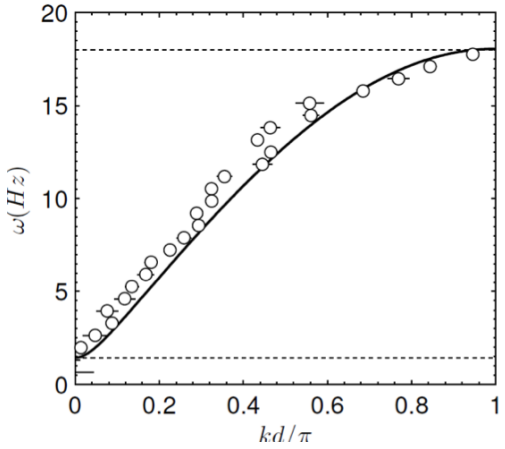
$$\frac{d^2u_n}{dt^2} + w_0^2 \sin u_n = - \frac{1}{(1+u_{n+1} - u_n)^2} + \frac{1}{(1+u_n - u_{n-1})^2}$$

Low-amplitude limit

$$\frac{d^2u_n}{dt^2} + w_0^2 u_n = k(u_{n+1} - 2u_n + u_{n-1}) - a(u_{n+1} - 2u_n + u_{n+1})(u_{n+1} - u_{n-1})$$

Dispersion relation

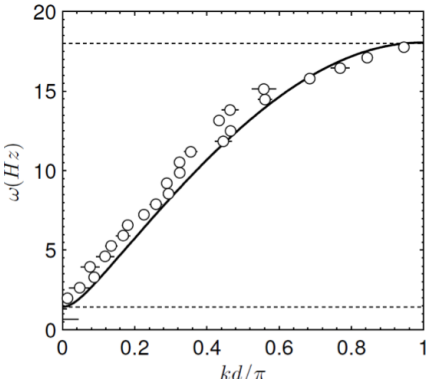
$$w^2 = w_0^2 + 4c_0^2 \sin^2 \left(\frac{ka}{2} \right)$$

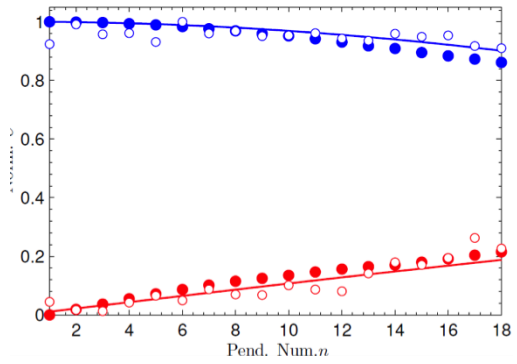


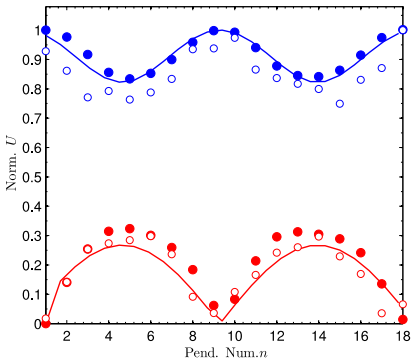
Theory (line)

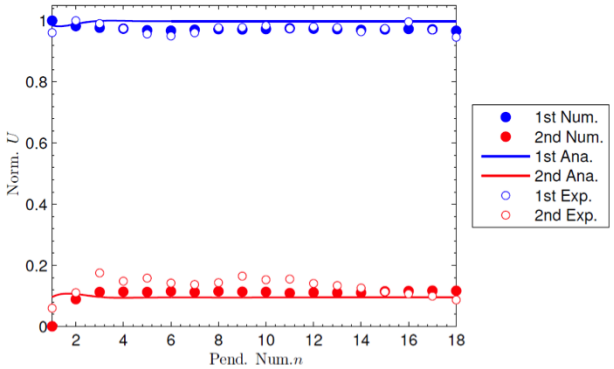
Experiment (symbols)

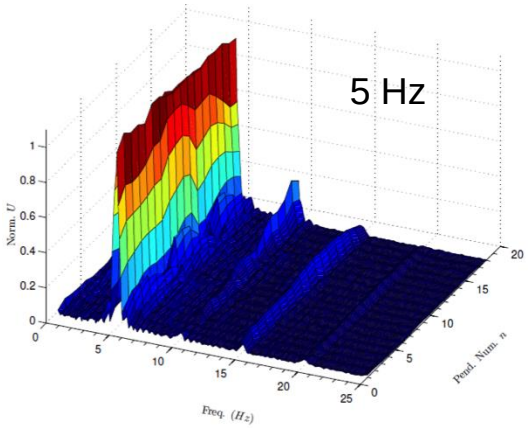
Harmonic generation in a Coulomb lattice

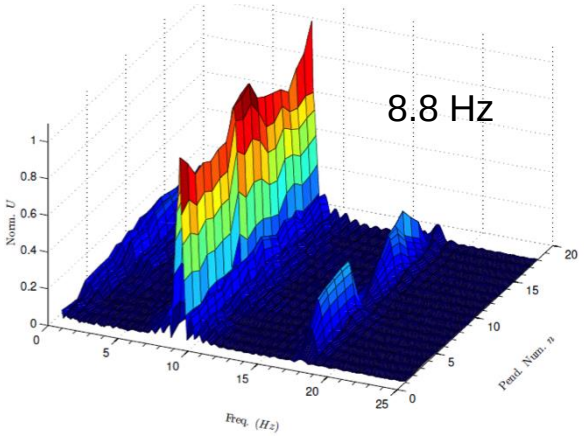


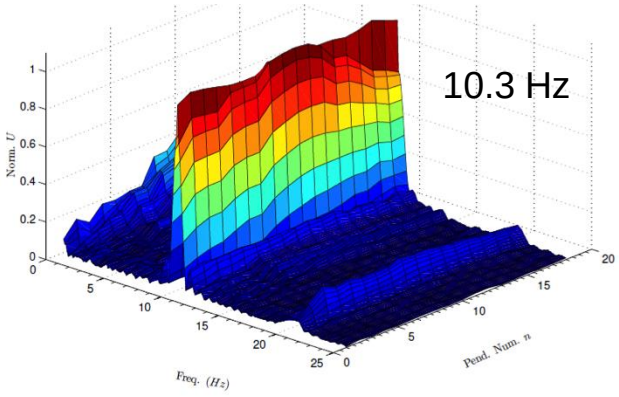




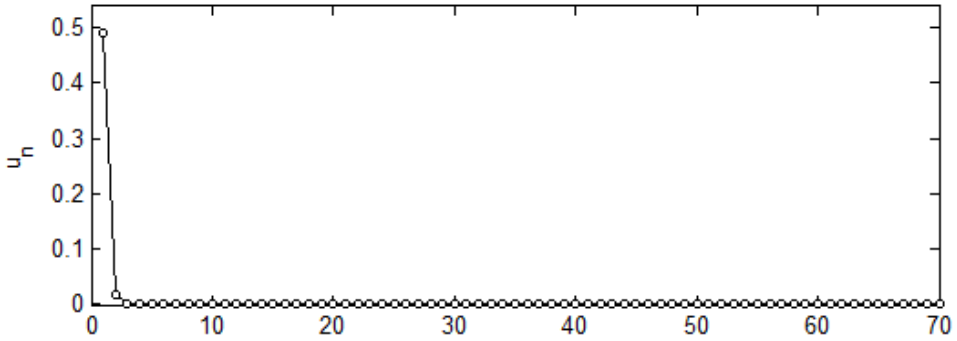






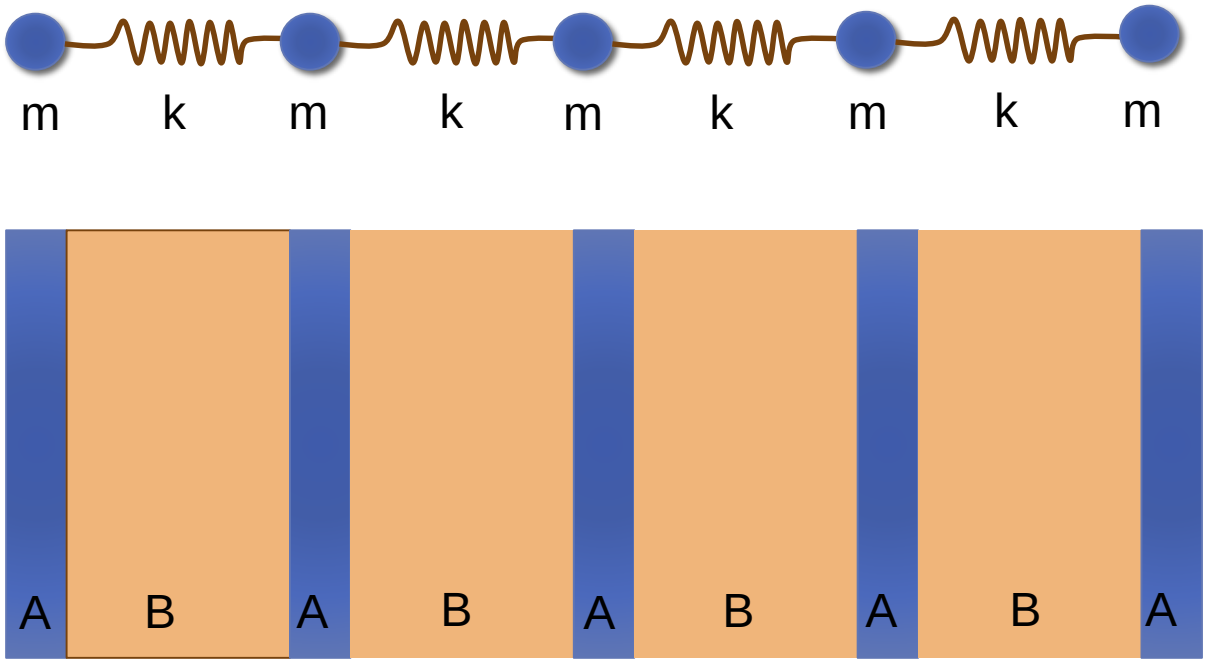


Localized modes: travelling kinks (solitons) and phonons in the lattice





A lattice-multilayer analogy



“Particle” layers: high mass density, low compressibility, thin

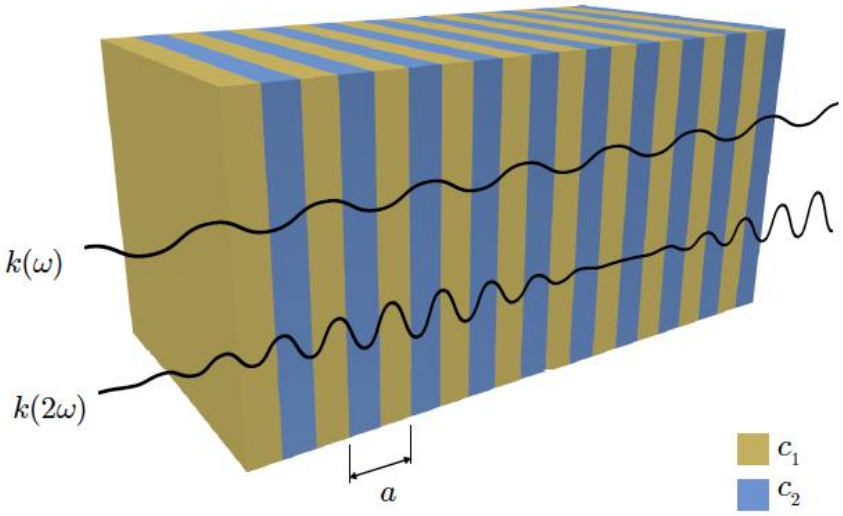
“Spring” layers: low mass density, high compressibility, thick

Some few conclusions about lattices

- Simple models like FPU can explain analytically the complex dynamics of harmonic generation
- Even for highly nonlinear lattices, like granular or coulomb type
- Monoatomic lattices (with single propagation band) can be extended to diatomic, etc, to get multiple bands and modes: more degrees of freedom

So next step: what new effects can be expected in multilayers?

1D crystals (multilayers, superlattices)

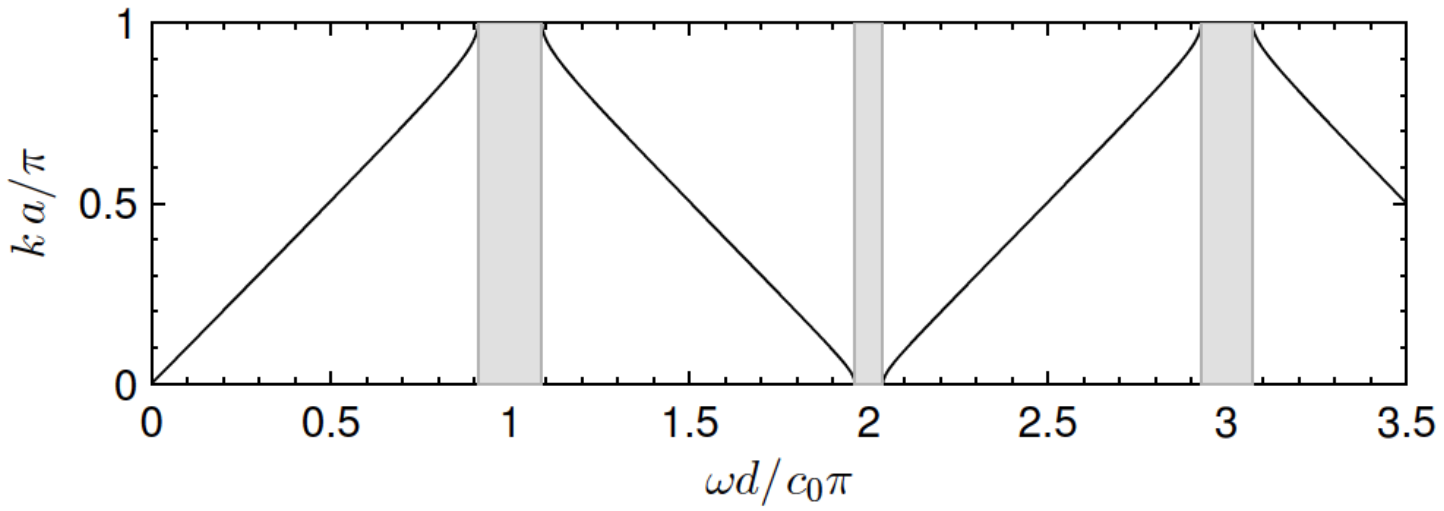


- The layers may differ in density, elasticity, nonlinearity,…
- May be fluid/fluid, fluid/solic or solid/solid
- Used as: phononic mirrors, phonon lasers,…
- Mostly studies in linear regime.

Dispersion and band structure

The dispersion relation (band structure) is **analytical**

$$\cos(kd) = \cos(k_1 d_1)\cos(k_2 d_2) - \frac{1}{2}\left(\frac{k_1}{k_2} - \frac{k_2}{k_1}\right)\sin(k_1 d_1)\sin(k_2 d_2)$$



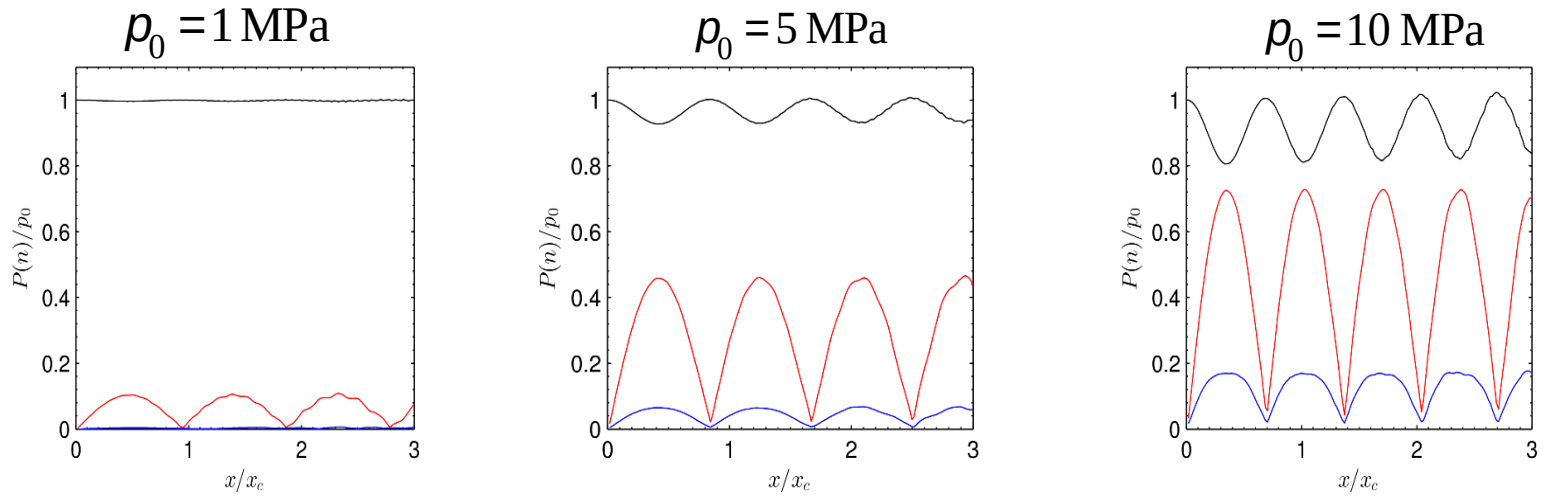
Nonlinear propagation effects

- Single-band (lattice) effects:
  - Asynchronous harmonics: Beatings. Nonlinear dispersion
  - Forbidden waves with finite amplitude in the bandgap
- Multi-band effects: Waveform management
  - All waves in synchrony: nondispersive high frequency waves
  - Enhancement of 2nd harmonic: optic-like case
  - Enhancement of 3rd harmonic: cubic-like nonlinearity
- Progressively changing bands: chirped crystals

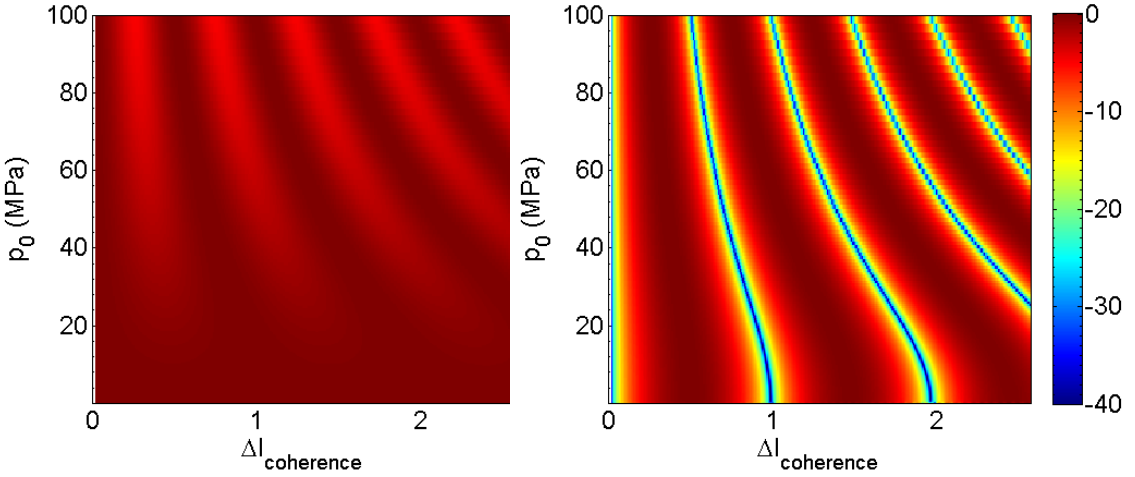
Numerical study by solving constitutive equations for (p, ρ, $\mathbf{v}$) using FDTD method

2nd harmonic: in-band case

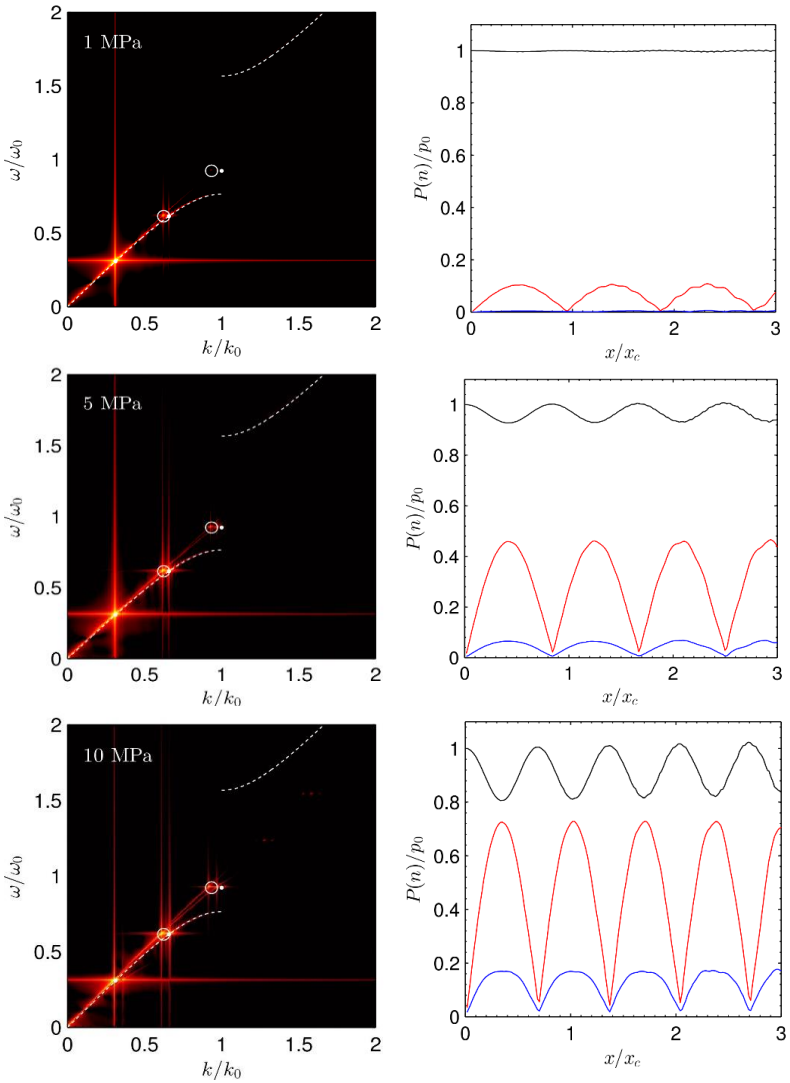
- **Dispersion:** beatings at the half period given by the coherence length

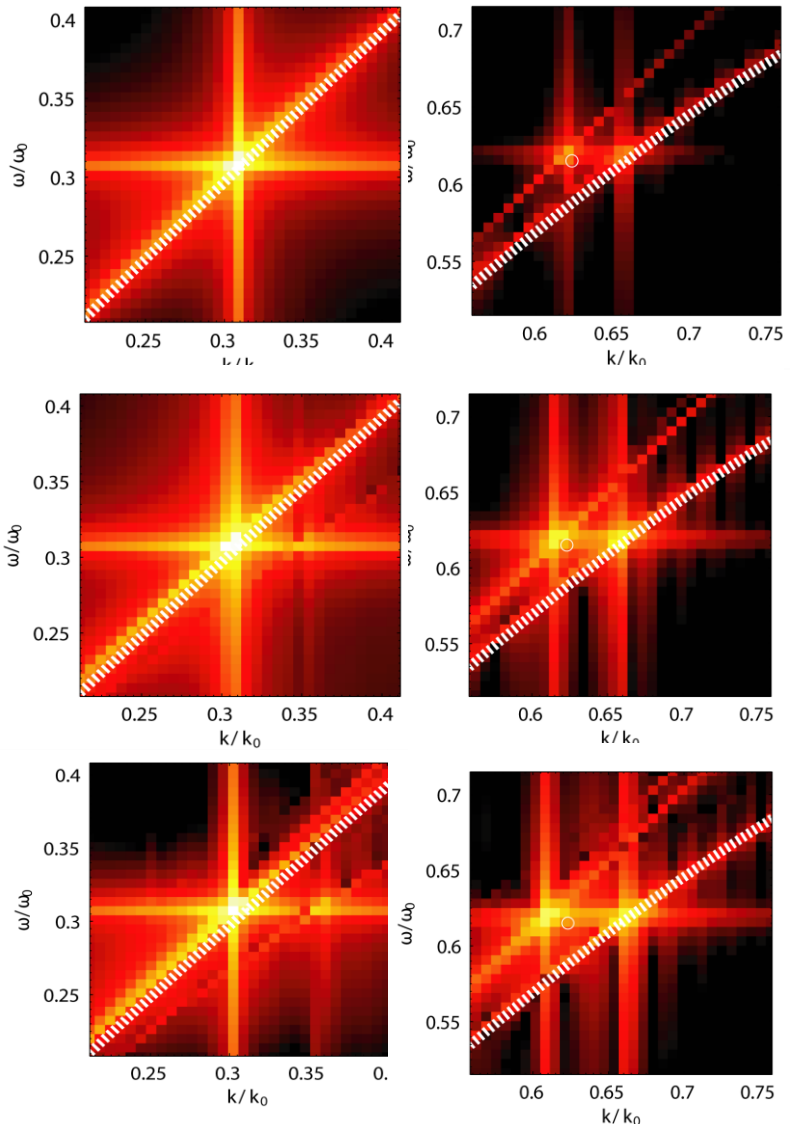


- The beating period depends o the amplitude
- Signature of **nonlinear dispersion**



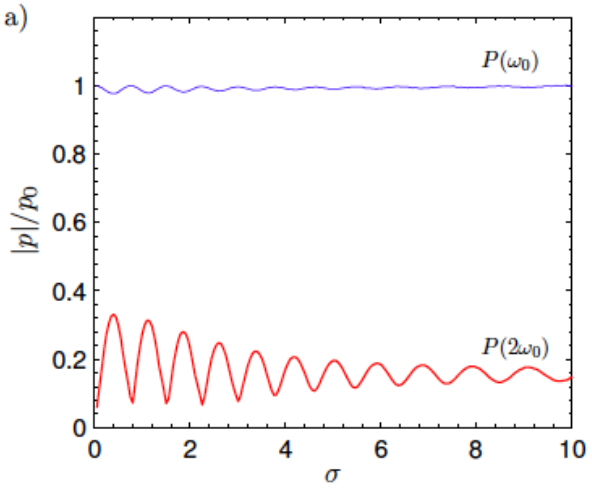
Nonlinear dispersion



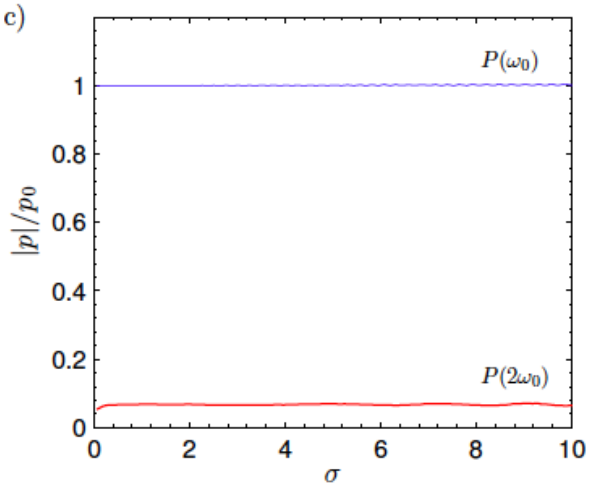


2nd harmonic: Bandgap case

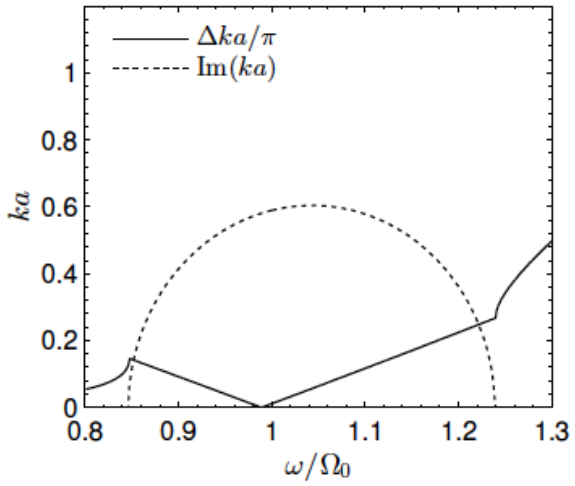
- Free (homogeneous) wave decay, but forced not!
- 2nd harmonic gets a finite (constant) amplitude



just above the BG



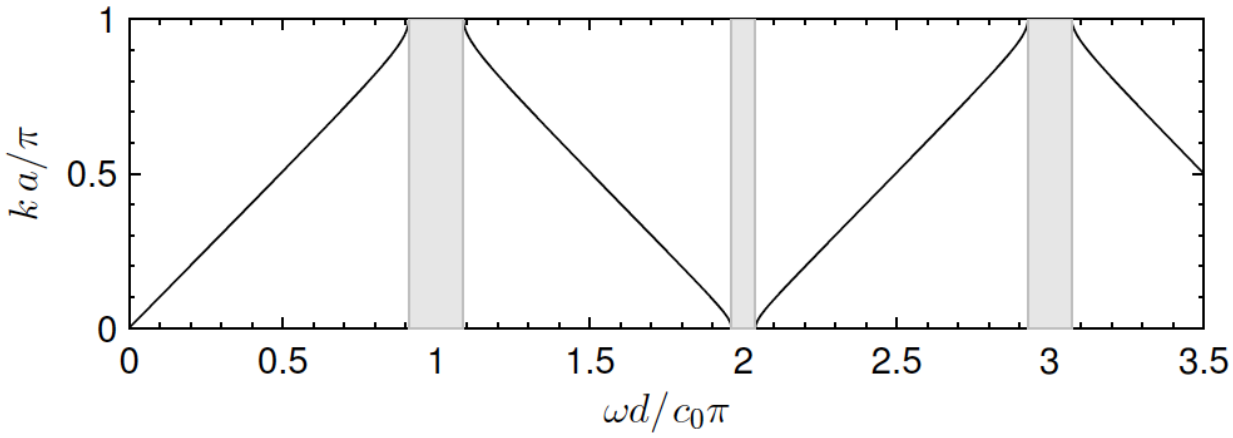
in the middle of the BG



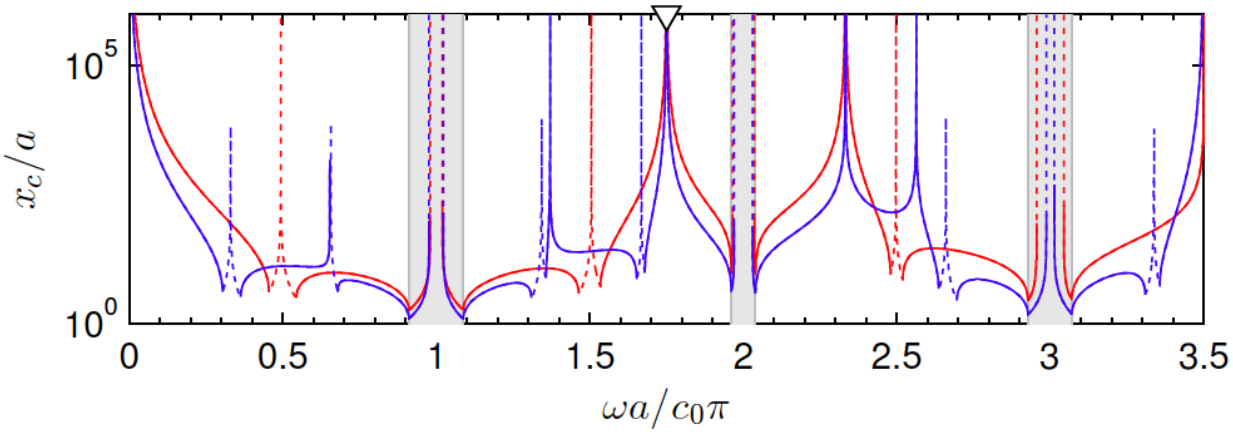
Detuning and attenuation vs frequency

Managing coherence between harmonics

- Dispersion relation may be used to evaluate coherence length



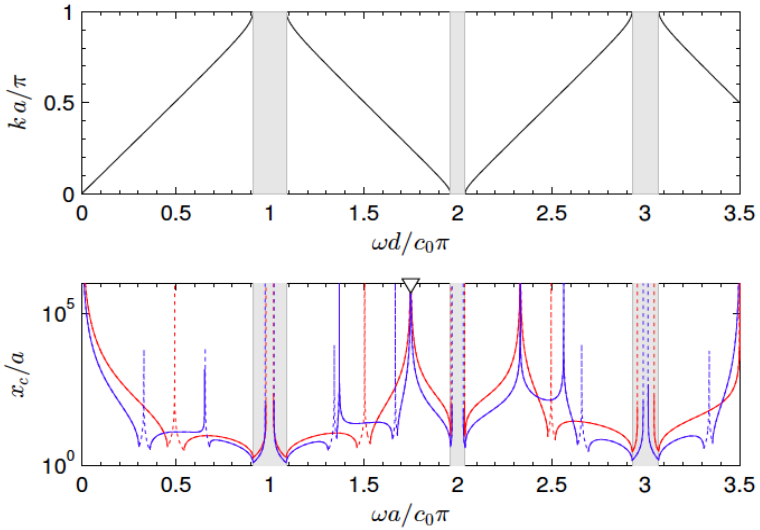
Dispersion relation

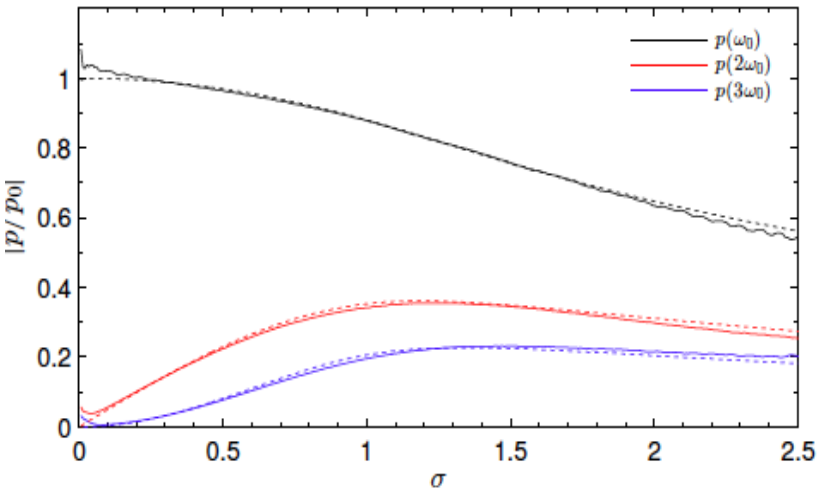


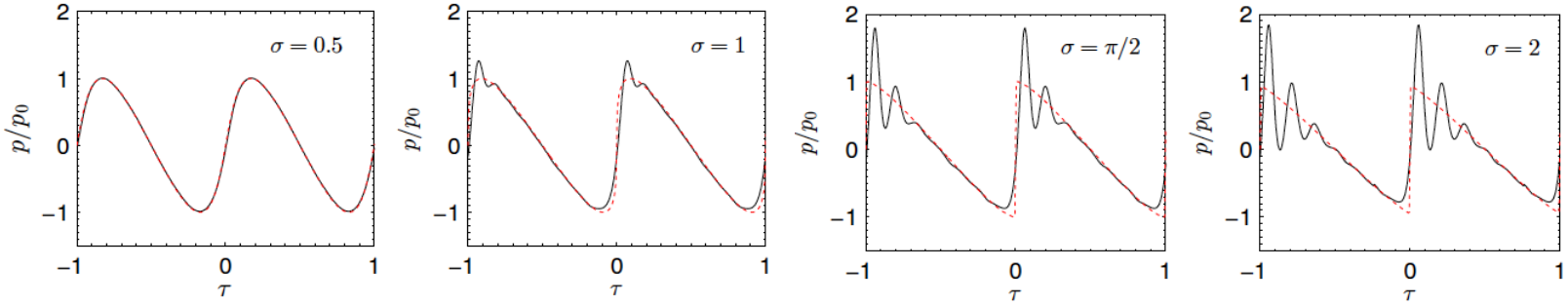
Coherence length:

Red: ω-2ω
Blue: ω-3ω

First three harmonics in synchrony: a quasi non-dispersive medium at high frequencies



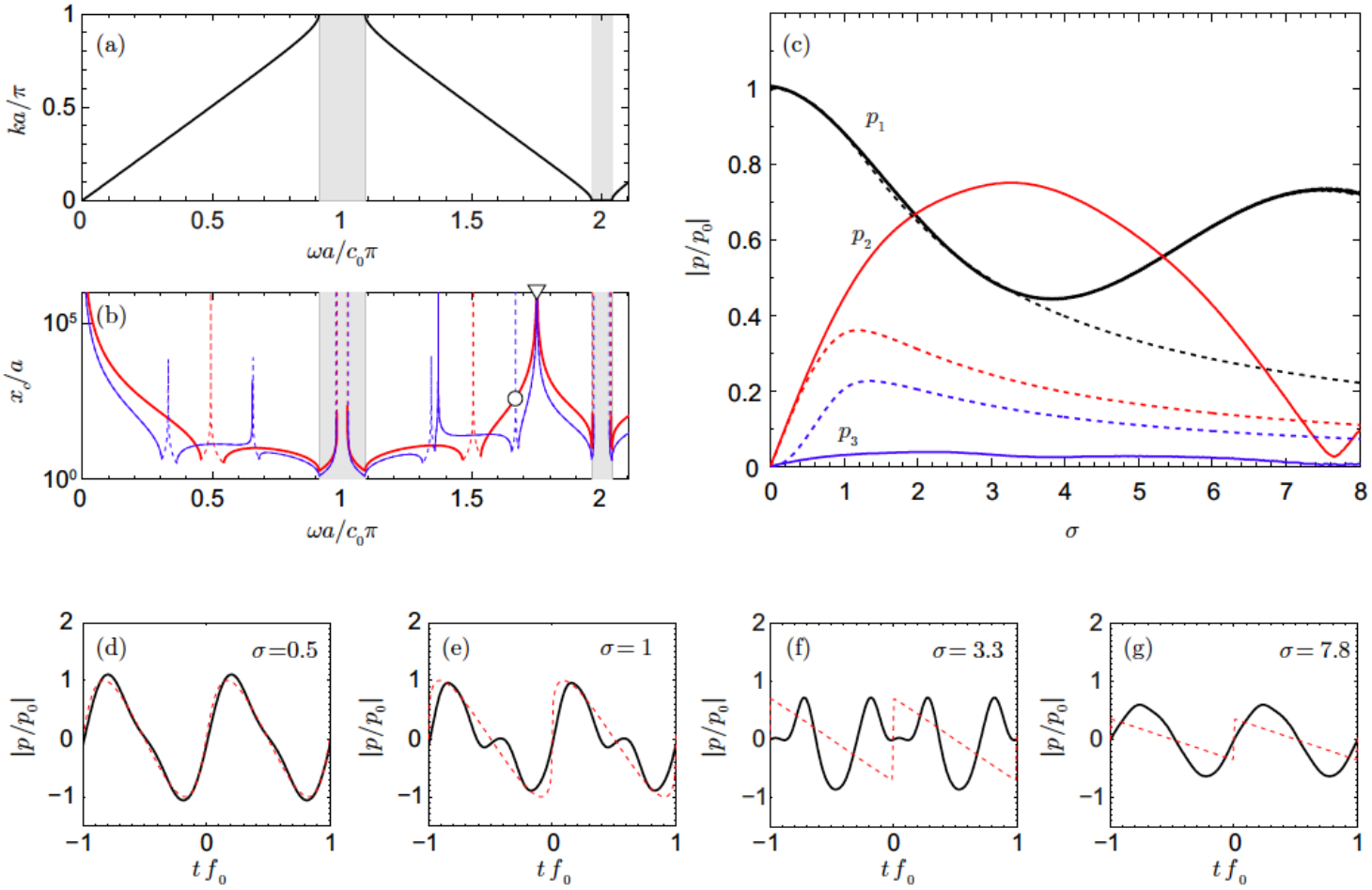




Enhancement of second harmonic

2nd harmonic with high coherence. 3rd harmonic in a BG.

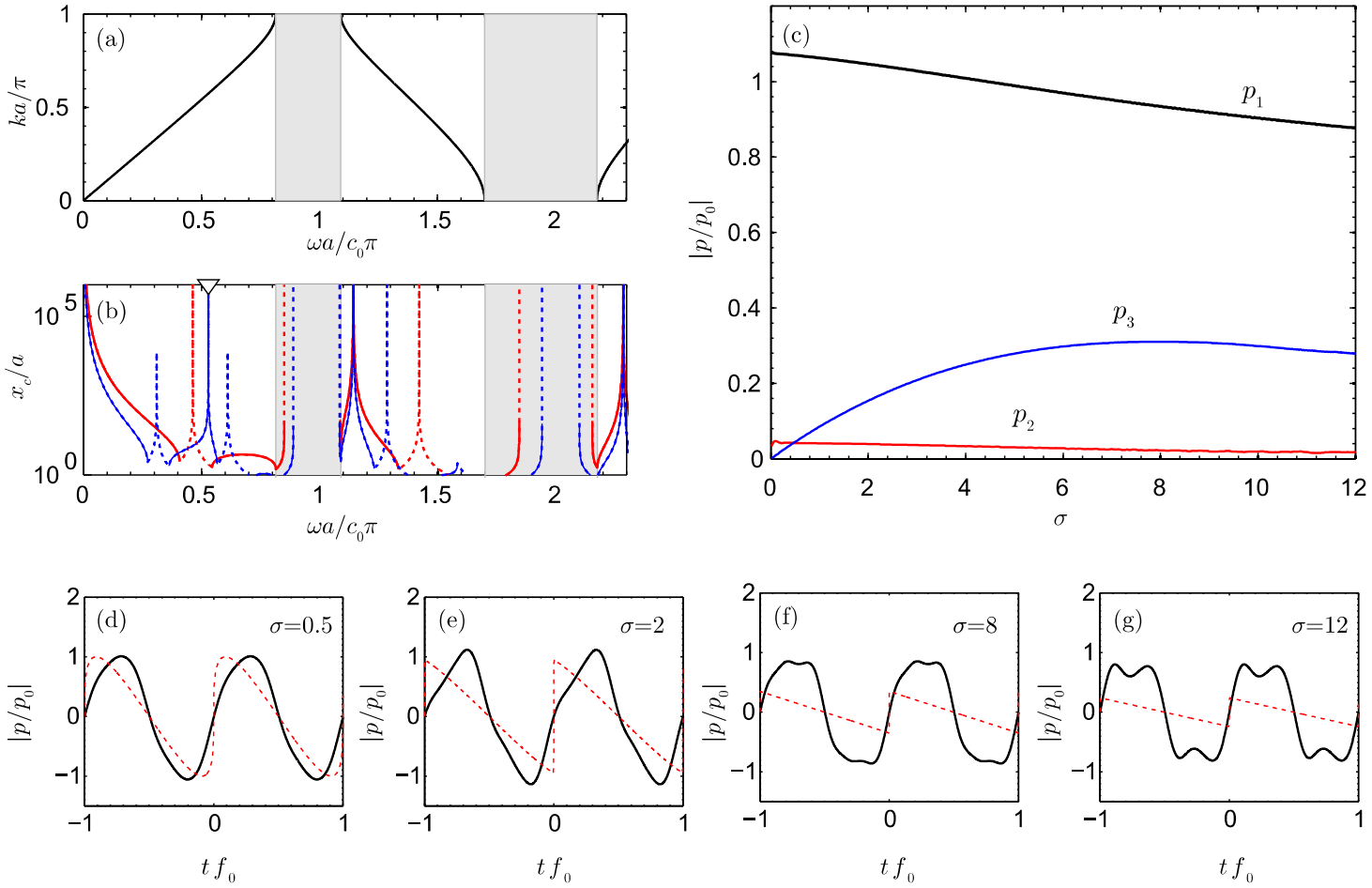
Second harmonic amplitude overcomes first harmonic!



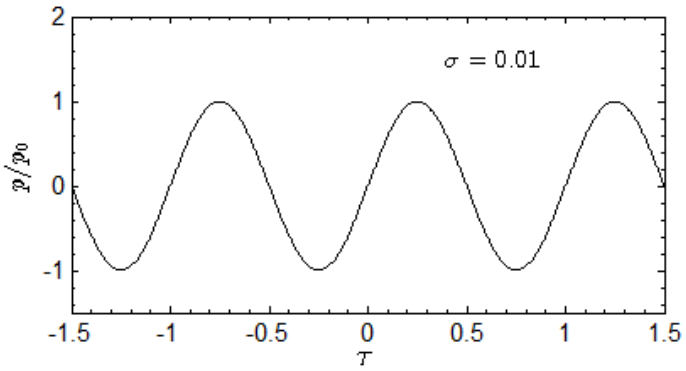
Enhancement of third harmonic

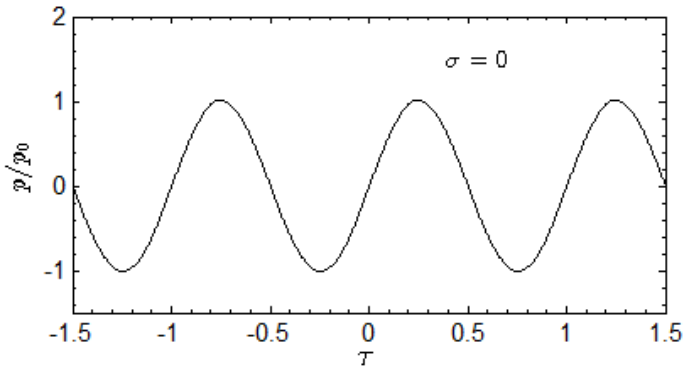
2nd harmonic fall in bandgap. Perfect coherence for the 3rd harmonic.

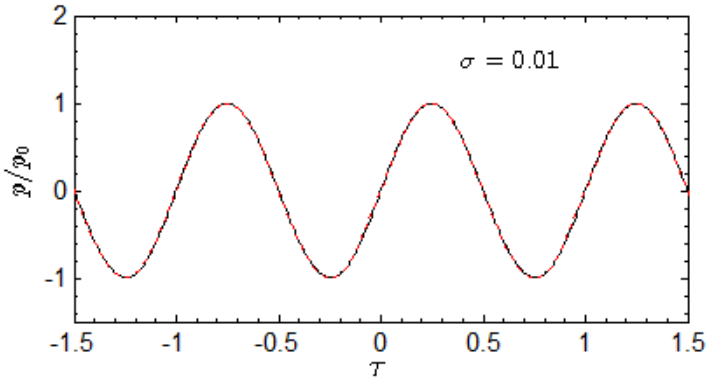
Result in an **effective cubic-like nonlinear medium**

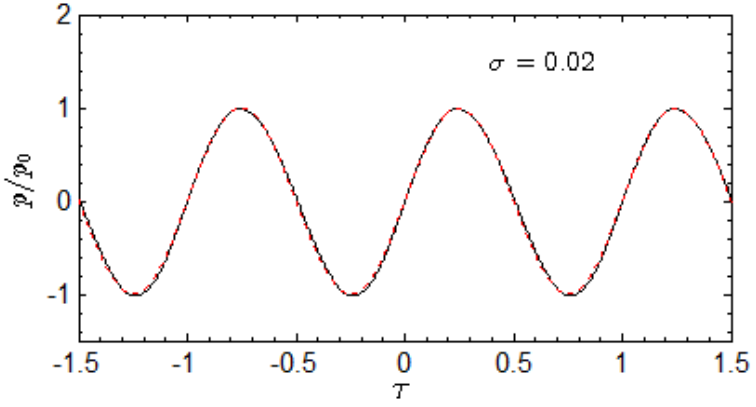


Time waveforms: comparison

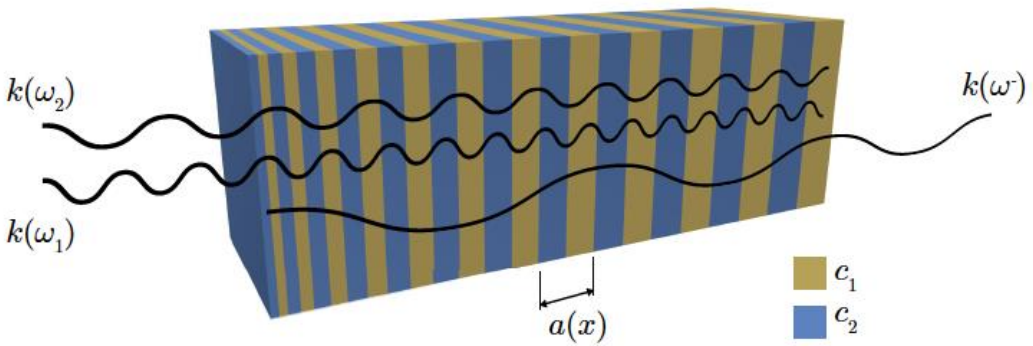


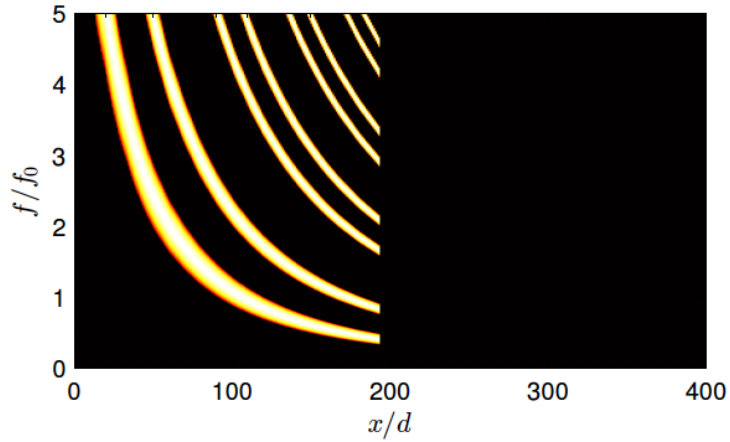


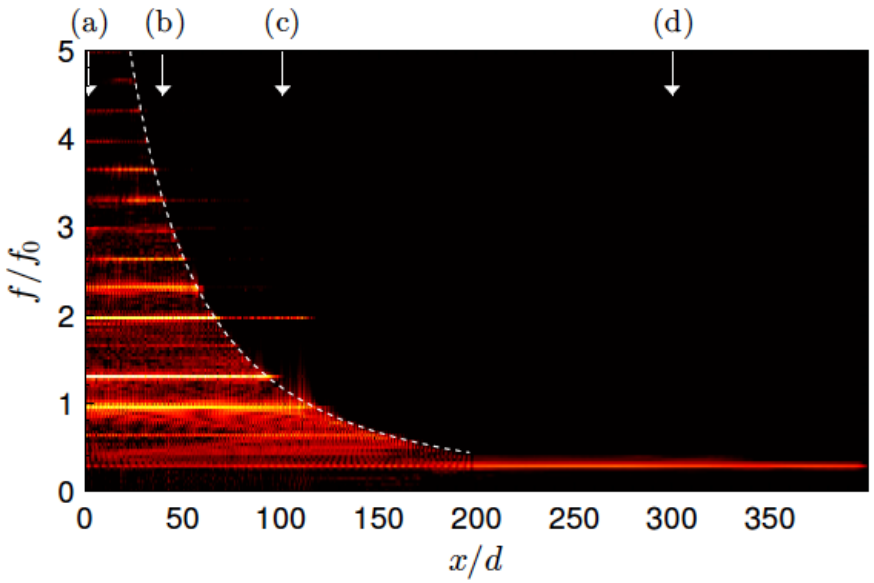


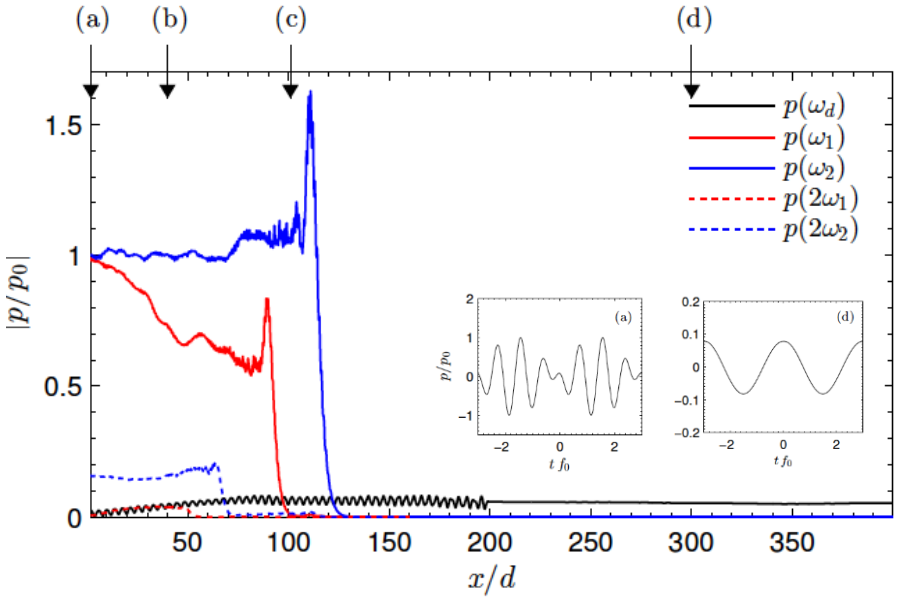


Chirped structures



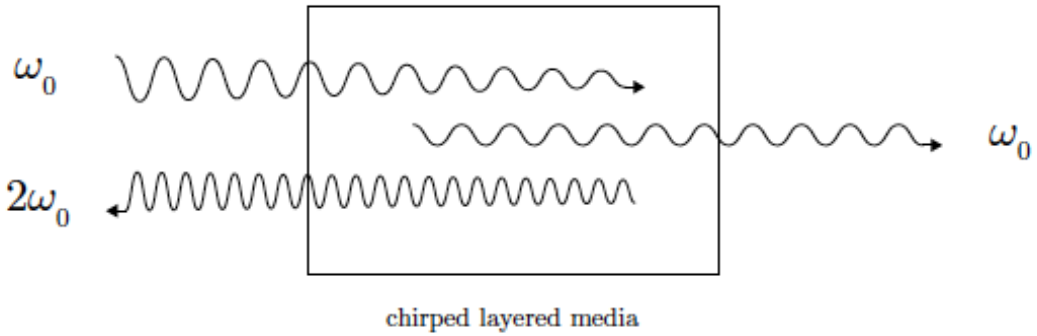


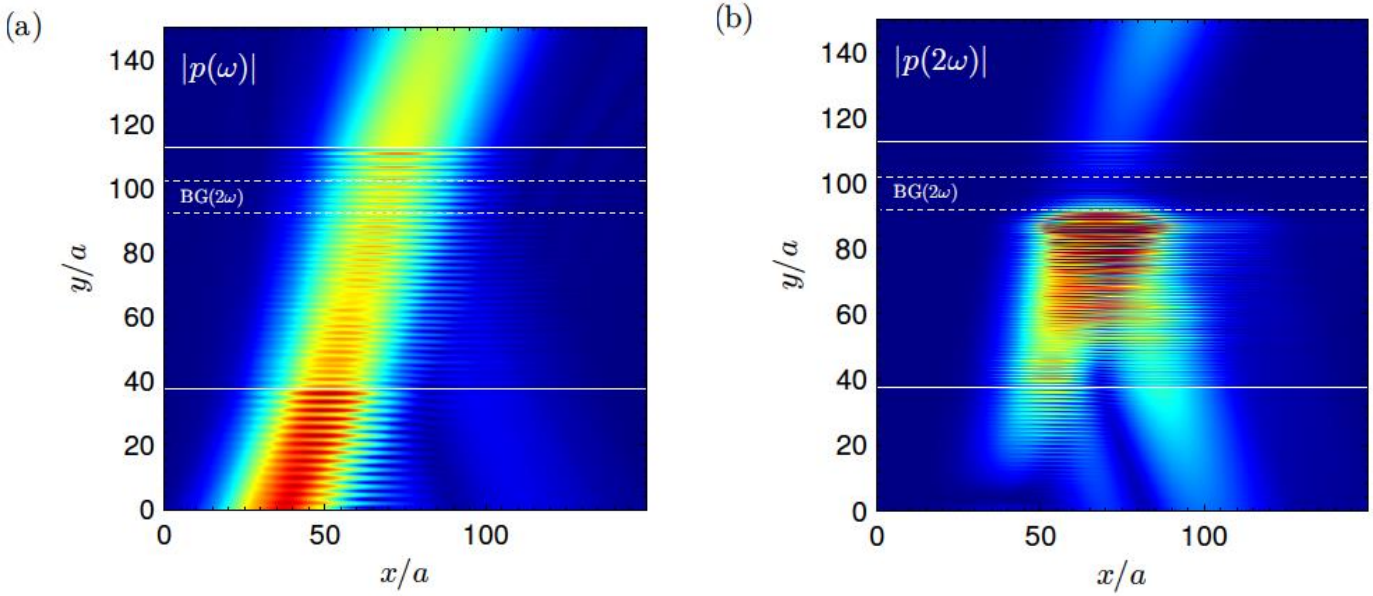




Frequency conversion mirror

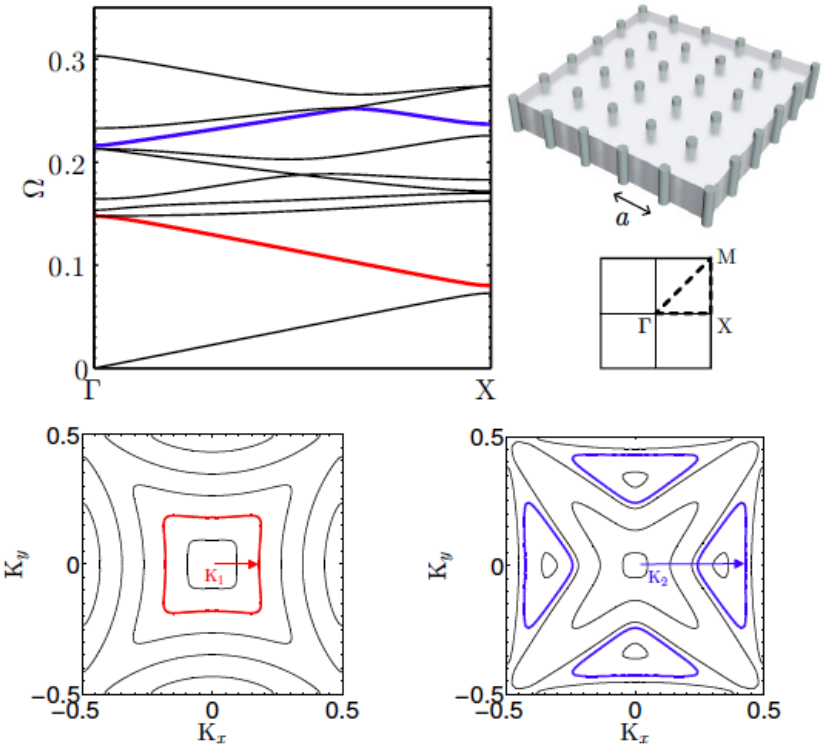
Chirped structure with BG for second harmonic but nor the first

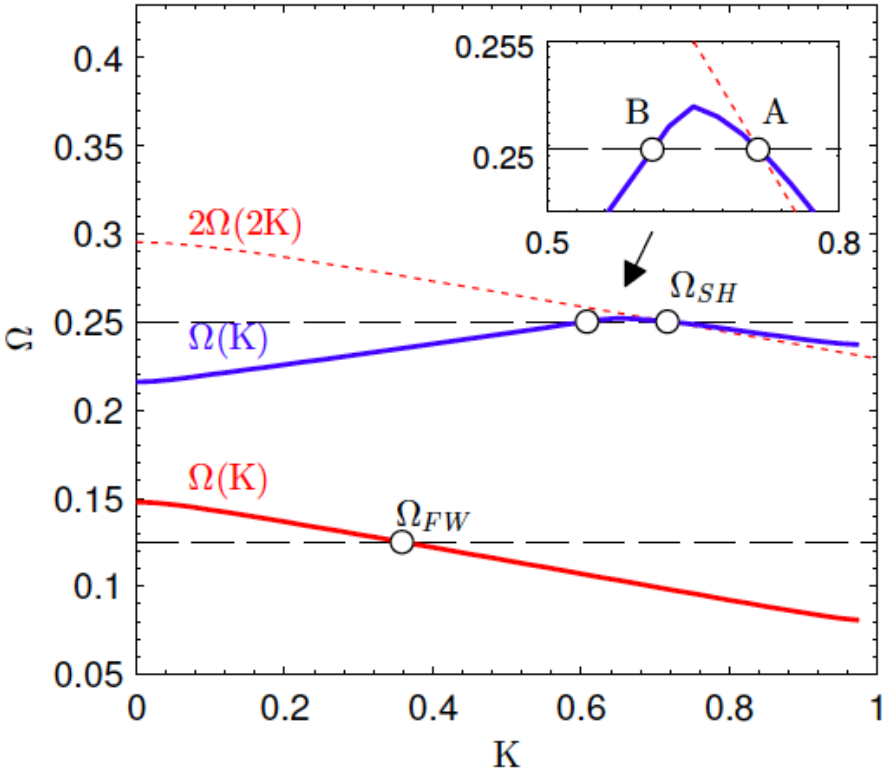




Beams in 2D crystals

Self-collimation of 2D beams

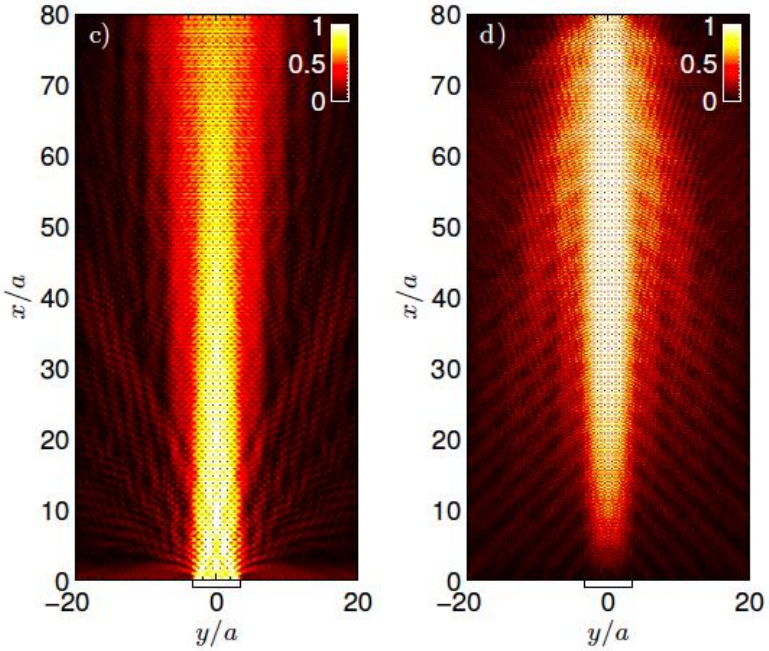


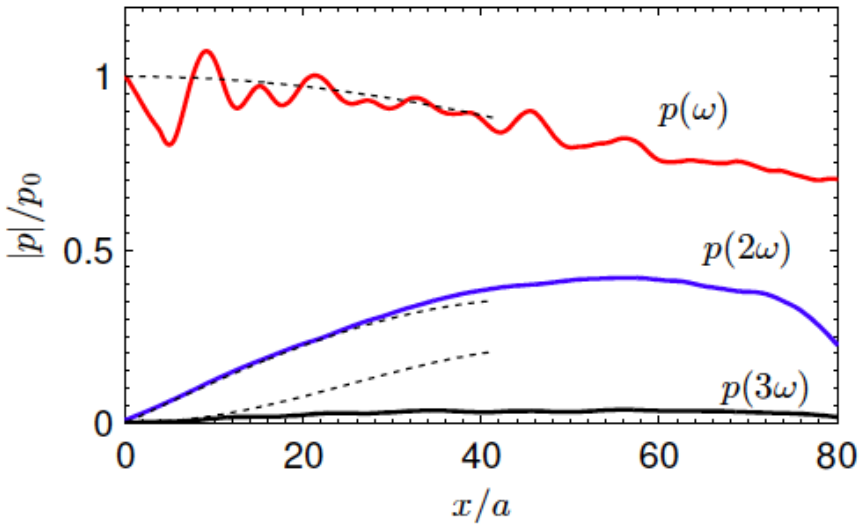


Efficient generation of 2nd harmonic 2D beams

First harmonic

Second harmonic





Conclusion

The interplay between nonlinearity and periodicity offers new and interesting possibilities to control wave propagation in lattices and periodic materials

1D systems allow for analytical predictions based on very simple models

In 2D many nonlinear effects for beams can be investigated

We have considered classical (weak, quadratic) nonlinearities. But there many other nonlinearity types to be exlored

Experiments to confirm the predictions are in progress

THANK YOU FOR YOUR ATTENTION

ADDITIONAL MATERIAL

A few of theory

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{1}{\rho_0}\nabla\rho_0\nabla p = -\frac{\beta}{\rho_0 c_0^4}\frac{\partial^2 p^2}{\partial t^2}$$

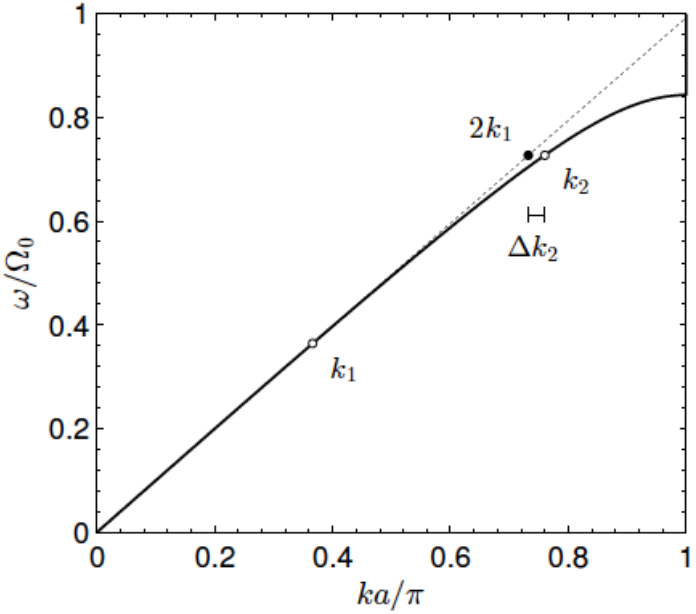
$$p = p^{(1)} + \varepsilon p^{(2)} + \cdots,$$

$$p^{(1)} = p_0 \sin(\omega t - k_1 x)$$

$$\frac{\partial^2 p^{(2)}}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p^{(2)}}{\partial t^2} = -\frac{4\beta\omega^2 p_0^2}{\rho_0 c_0^4}\sin(2\omega t - 2k_1 x)\,.$$

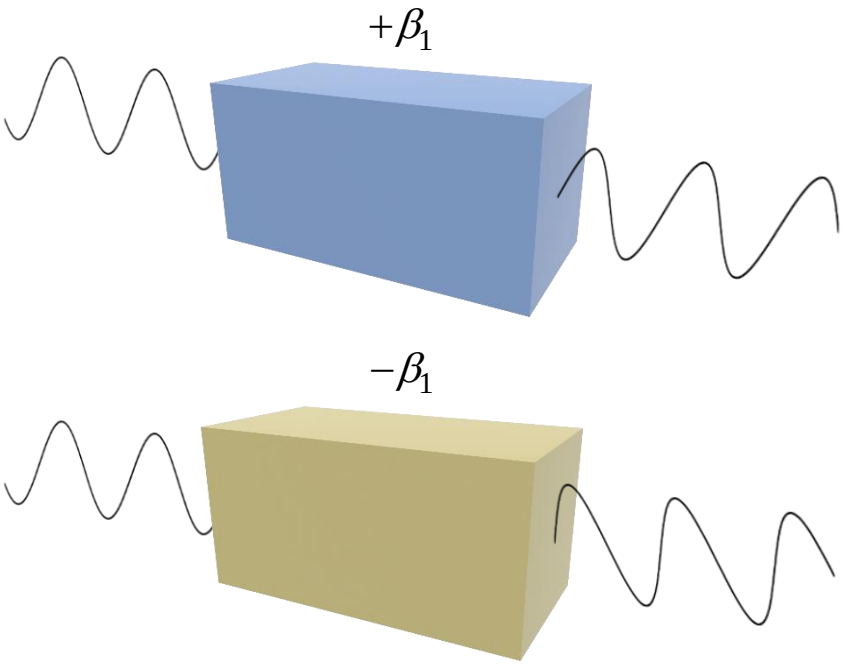
$$p^{(2)} = p_h^{(2)} + p_f^{(2)}$$

$$x_c = \frac{\pi}{|\Delta k|} = \frac{\pi}{|k(2\omega) - 2k(\omega)|}$$

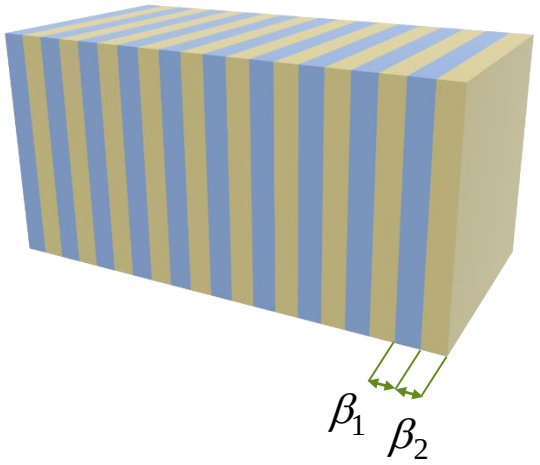


Modulated nonlinearity

- Alternate sign of nonlinearity $c_1 = c_2 \quad a = b$



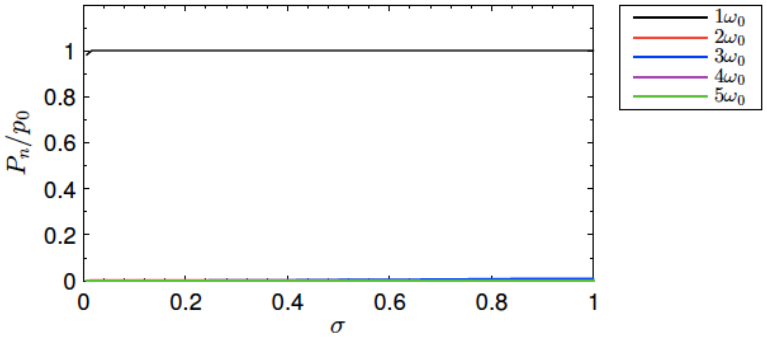
$\beta_2 \approx -\beta_1$

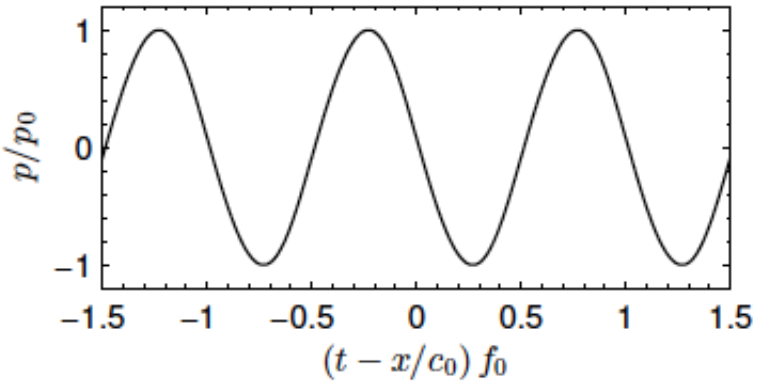


Distorsion compensation

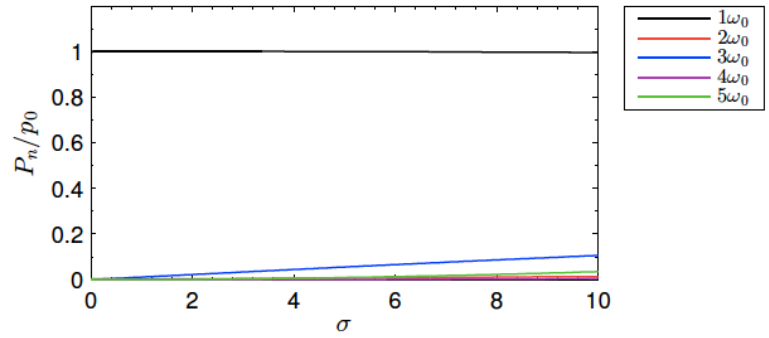
- Nonlinear effects are compensated for $x \sim \sigma$

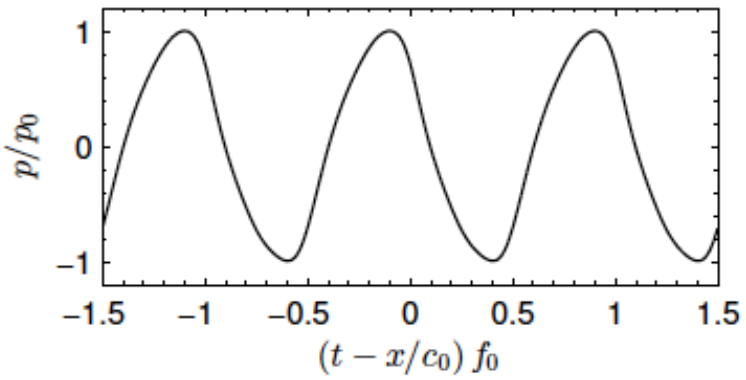
  An extraordinarily linear medium!





- For σ>>1, cubic nonlinear effects appear



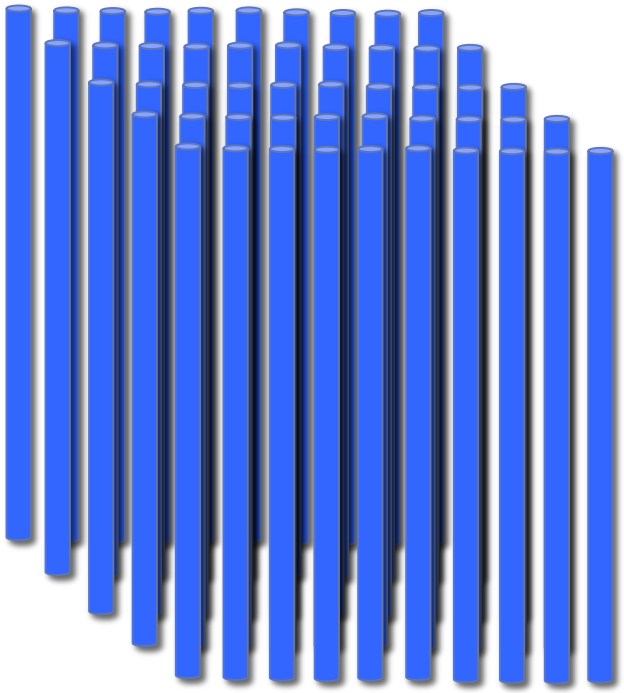


2D nonlinear sonic crystals

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P$$

$$p = c_0^2 \rho' + \frac{c_0^2}{\rho_0} \frac{B}{2A} \rho'^2$$

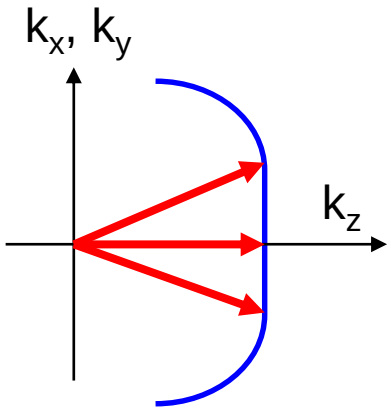


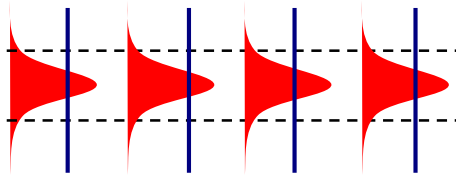
- Scatterers are considered as rigid
- Nonlinearity only in the host medium (fluid)

Self-collimation - review

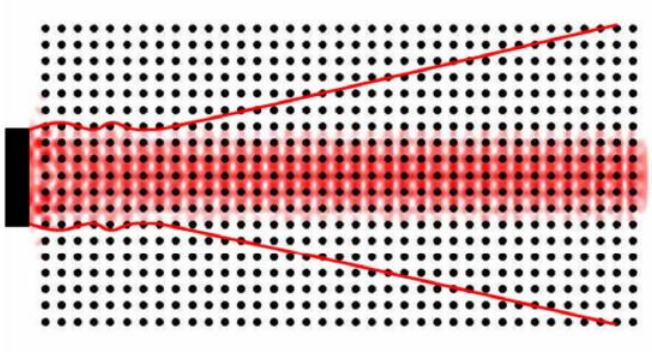
Self-collimation:

- Propagation of PWs is perpendicular to the IFC
- A flat IFC results in an effective zero diffraction

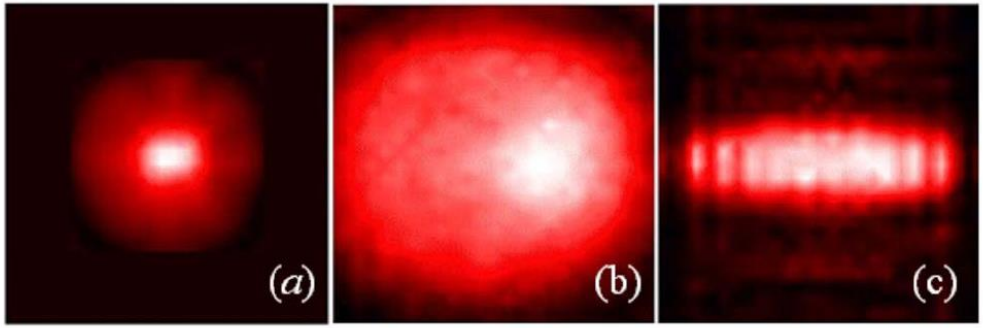




Numerical simulation



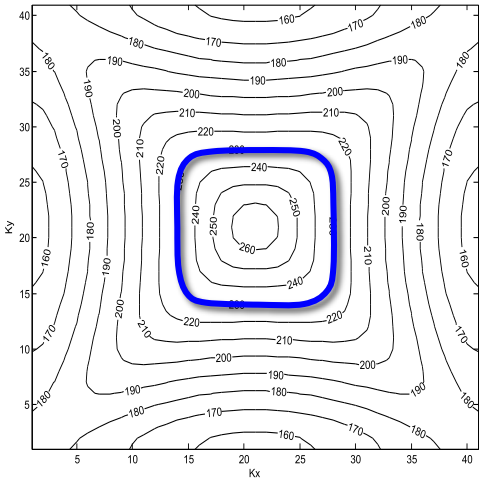
Experimental results



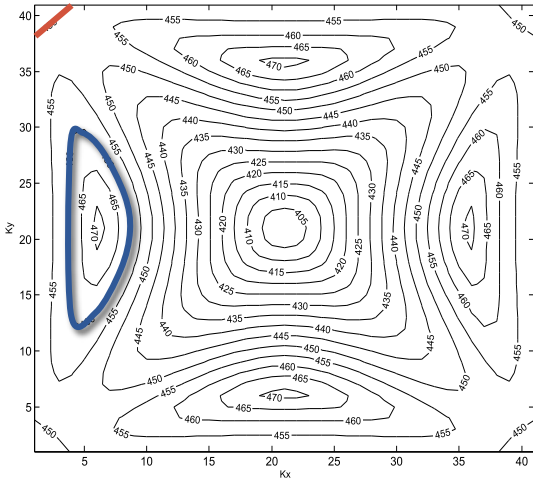
V. Espinosa et al, PRB **76**, 140302R 2007

2D nonlinear sonic crystals

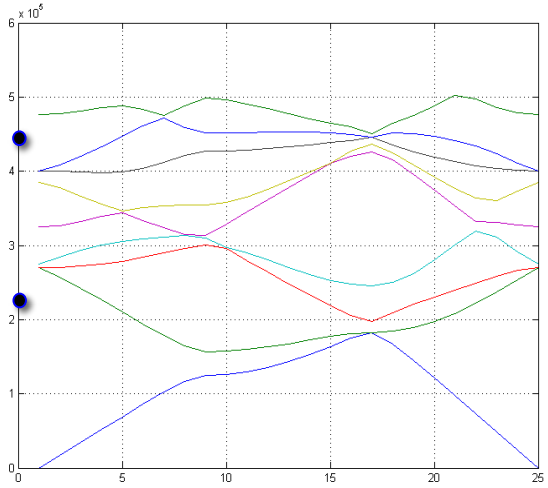
Dispersion curves at self-collimation condition



Second band

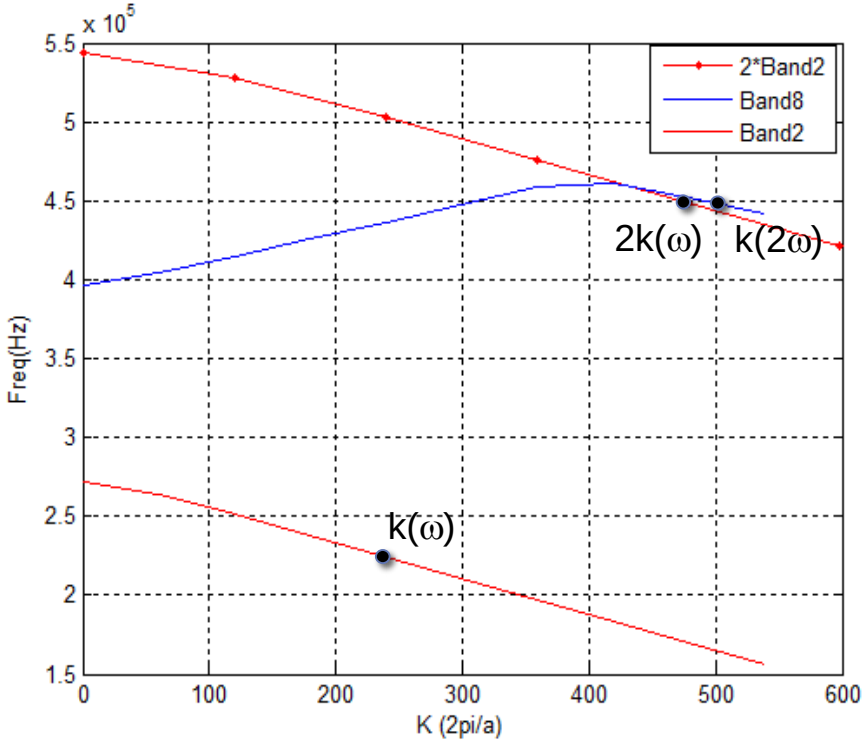


Eigth band

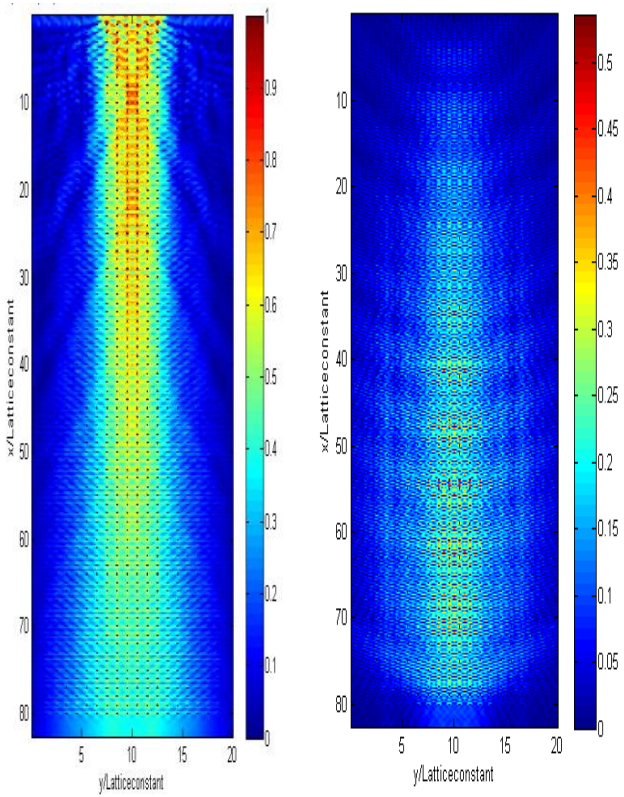


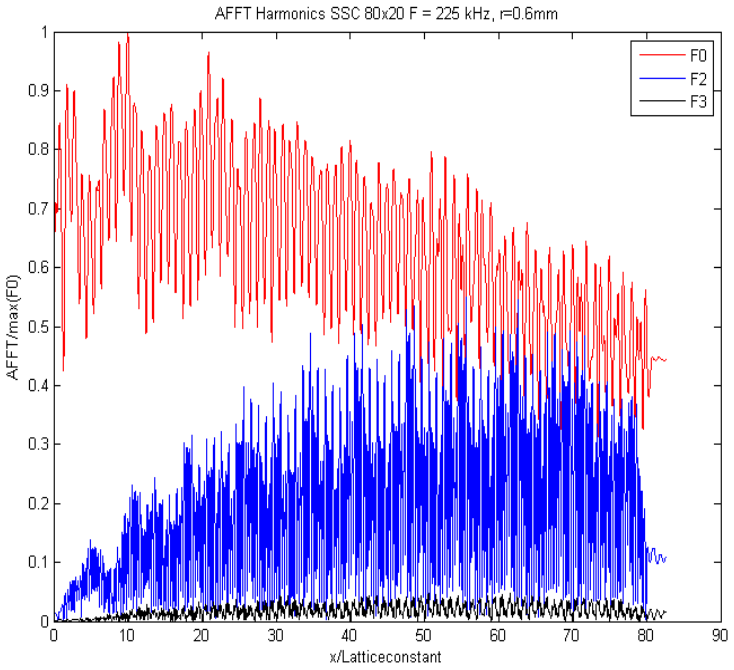
Isofrequency curves and band structure of the crystal:
r = 1mm, a = 5.25mm, host medium: water.

Phase matching of second harmonics



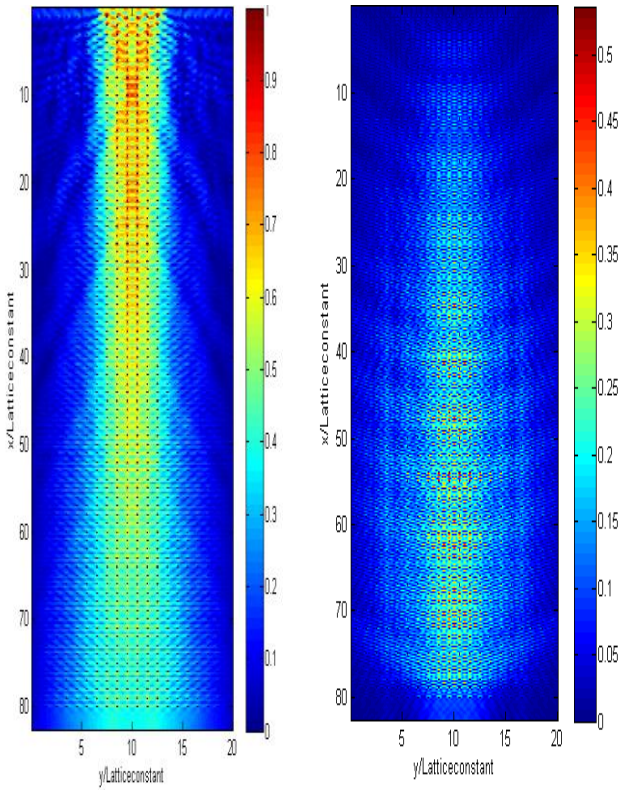
2D nonlinear sonic crystals

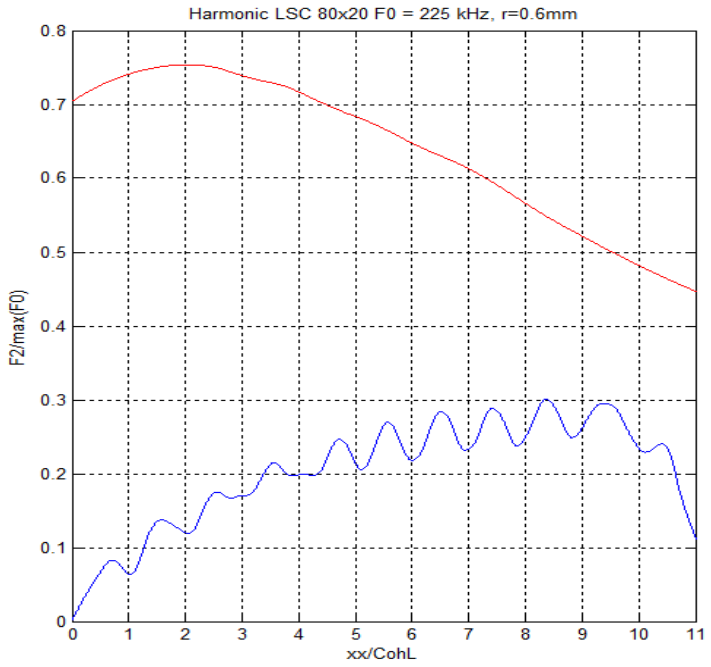




r = 0.6mm, a = 5.25mm, frequency (Fundamental) = 225 kHz, P0 = 1.5MPa, circular scatterers, Transducer diameter Ra = 35mm

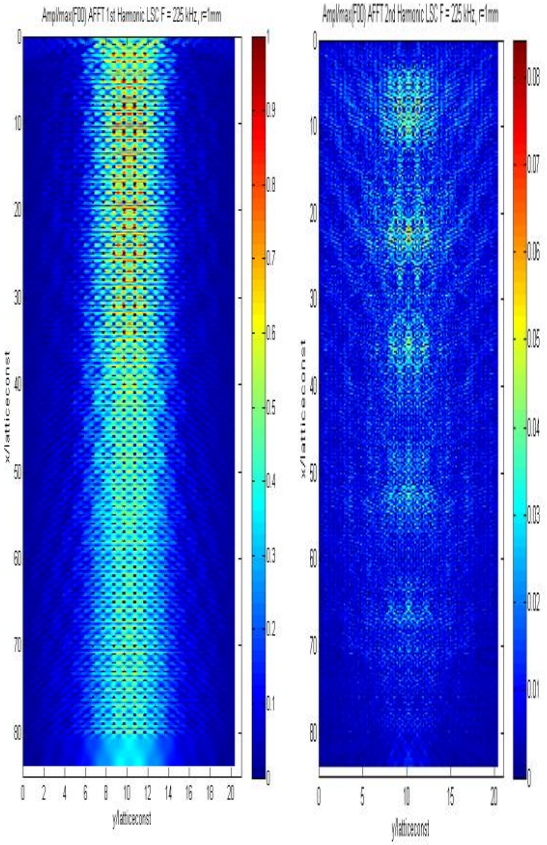
2D nonlinear sonic crystals

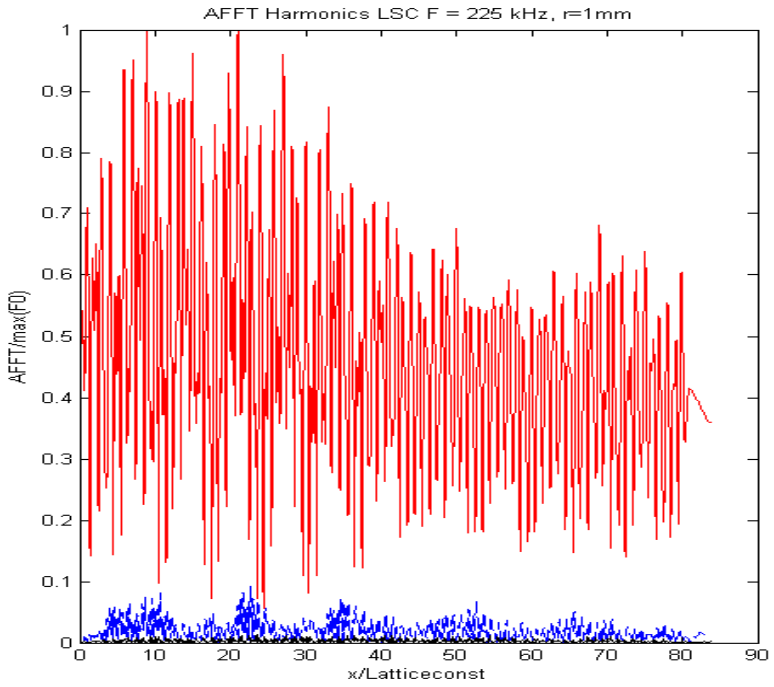




r = 0.6mm, a = 5.25mm, frequency (Fundamental) = 225 kHz, P0 = 1.5MPa, circular scatterers, Transducer diameter Ra = 35mm

2D nonlinear sonic crystals





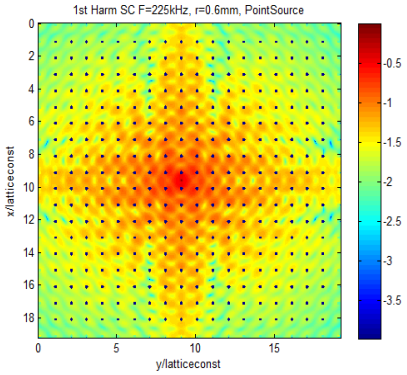
r = 1 mm, a = 5.25 mm, frequency (Fundamental harmonic) = 225 kHz, P0 = 1.5MPa, circular scatterers, Transducer diameter Ra = yy/3 = 35mm

2D nonlinear sonic crystals – point source

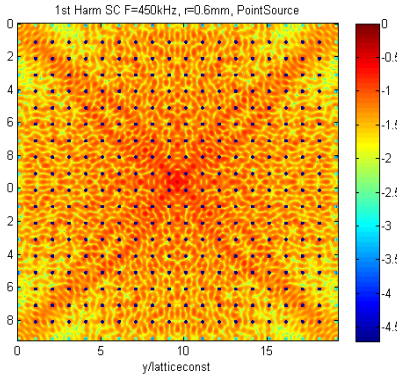
Point source at the center of the crystal

r=0.6 mm

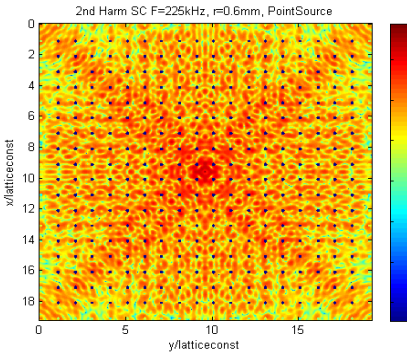
1st harmonic



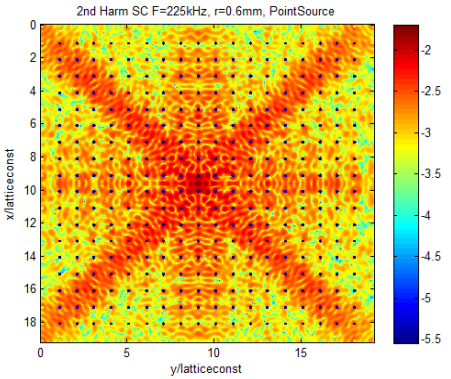
2nd harmonic
Linear



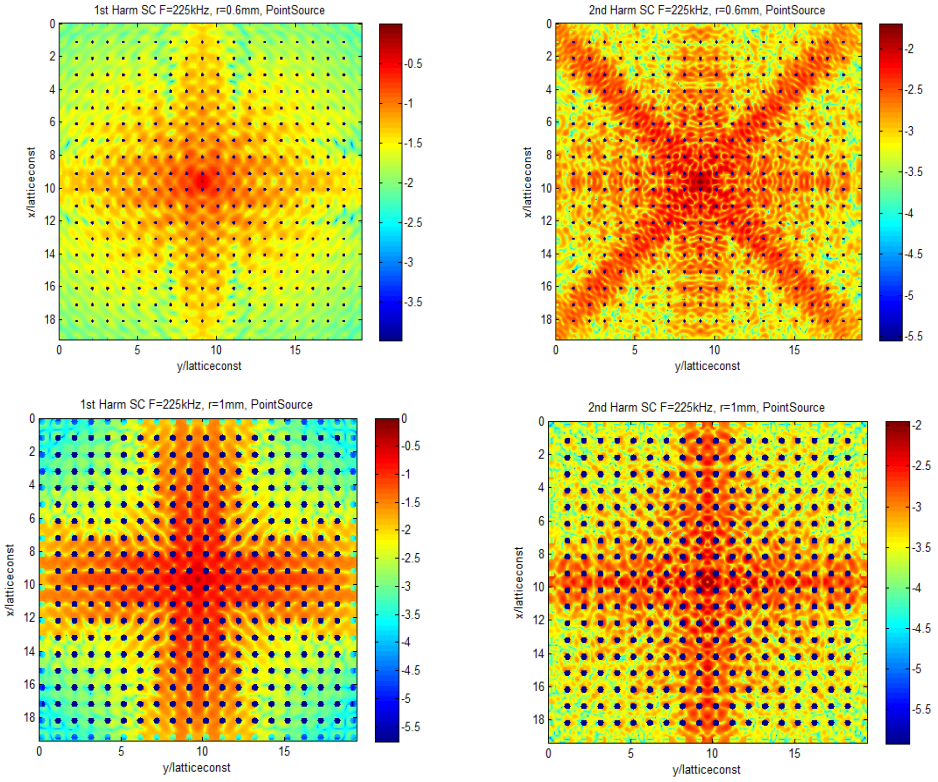
2nd harmonic
Nonlinear, weak



2nd harmonic
Nonlinear, strong



2D nonlinear sonic crystals



r=0.6mm

r=1mm

- Increasing filling factor modifies propagation directions (dispersion relations change)
- Since dispersion relations change with nonlinearity, it is possible to select the directions by varying the amplitude (nonlinear spatial filter)