NONLINEAR WAVES IN PHONONIC ROTATIONAL LATTICES

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Outline

- Highly nonlinear chain of coupled rotators
- Derivation of the model
- Analytical results: dispersion relations
- Numerical simulations
- Experiments

THEORY

The physical system





Equation of motion

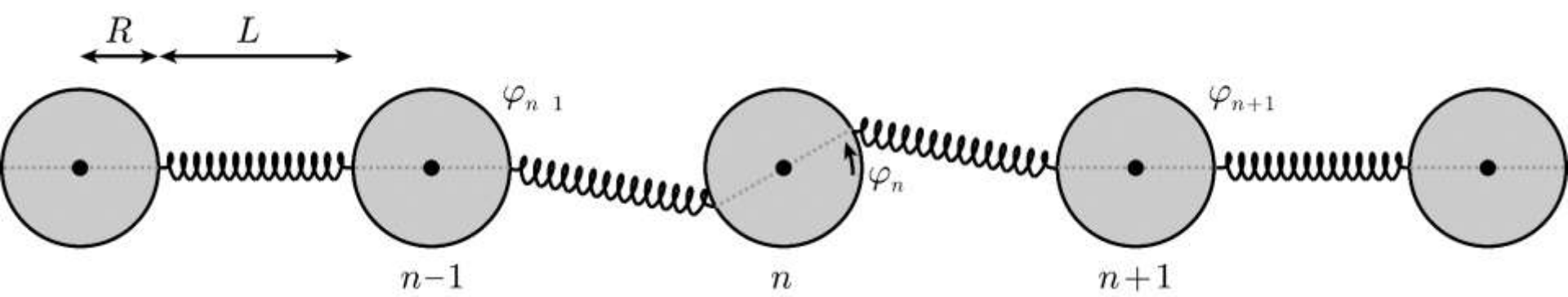
$$I\ddot{\phi}_n = M_{n-1,n} + M_{n,n+1}$$

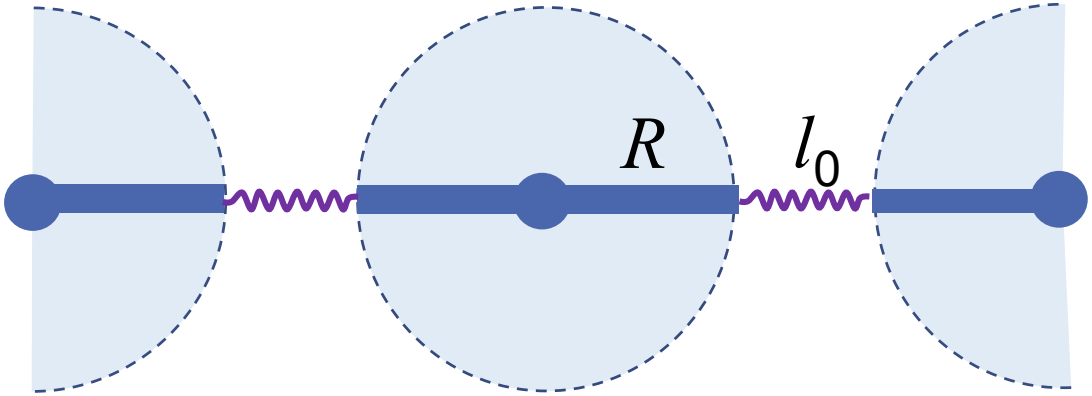
Moments

$$M_{n-1,n} = h_{n,n+1} F_{n,n+1}$$

Forces

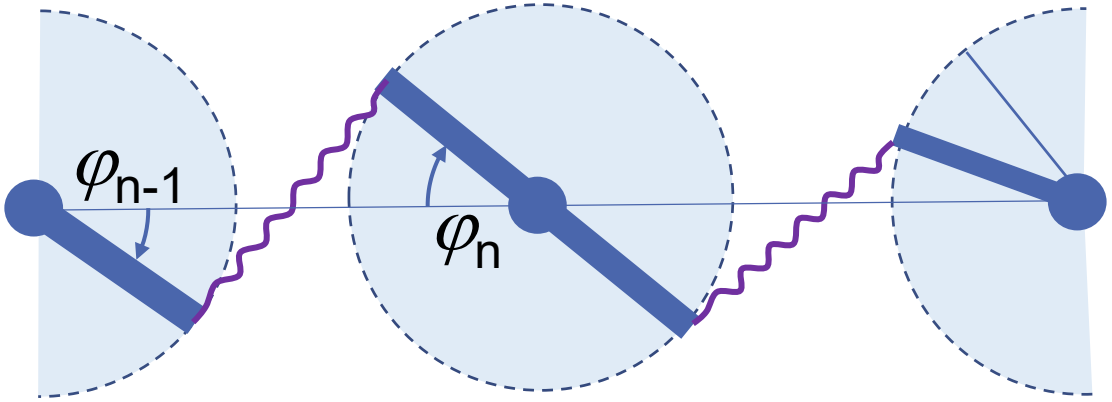
$$\left|F_{n-1,n}\right| = \eta \Delta l_{n-1,n}$$





No rotation (rest state)

No pre-stress



Rotation (excited state)

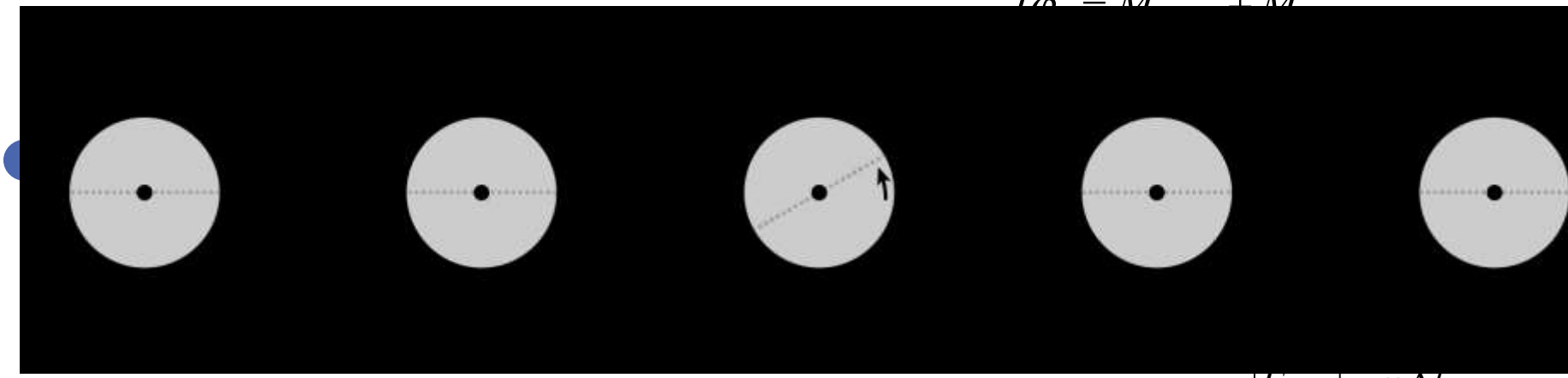
Stressed (linear)springs

Geometrical nonlinearity

Model derivation

Equation of motion

$$I\ddot{\phi}_n = M_{n-1,n} + M_{n,n+1}$$



$$|F_{n-1,n}| = \eta \Delta l_{n-1,n}$$

length of spring $$l_{n-1,n} = \sqrt{\left[l_0 + 2R\sin^2\left(\frac{\phi_n}{2}\right) + 2R\sin^2\left(\frac{\phi_{n-1}}{2}\right)\right]^2 + \left[R\left(\sin\phi_n + \sin\phi_{n-1}\right)\right]^2}$$

elongation (for small angle) $$\Delta l_{n-1,n} = l_{n-1,n} - l_0 \approx \frac{R}{4}\left[\left(1+\frac{2R}{l_0}\right)\left(\phi_{n-1}+\phi_n\right)^2 + \left(\phi_{n-1}-\phi_n\right)^2\right]$$

arm of the force $$h_{n-1,n} = R\left[\phi_n + \frac{R}{l_0}\left(\phi_{n-1}+\phi_n\right)\right]$$

Model

$$\ddot{\phi}_n = -\left(\frac{1-\mu^2}{12}\right)\left[\phi_{n-1}^3 + 2a\phi_n^3 + \phi_{n+1}^3 + 3\phi_n^2\left(\phi_{n-1}+\phi_{n+1}\right) + 3b\phi_n\left(\phi_{n-1}^2+\phi_{n+1}^2\right)\right]$$

$$\mu = \frac{l_0}{l_0+2R}, \quad a = \frac{1+\mu}{1-\mu}, b = \frac{1+\mu}{1-\mu} - \frac{8}{3}\frac{\mu}{1-\mu^2}$$

Geometrical parameters

$$\tau = \omega_0 t$$

$$\omega_0 = \sqrt{\frac{3\eta}{2I}}R\left(\frac{2R+l_0}{l_0}\right)$$

Typical frequency

Limiting cases

1. General case: for arbitrary geometry

$$\ddot{\varphi}_n = -\left(\frac{1-\mu^2}{12}\right)\left[\varphi_{n-1}^3 + 2a\varphi_n^3 + \varphi_{n+1}^3 + 3\varphi_n^2\left(\varphi_{n-1}+\varphi_{n+1}\right) + 3b\varphi_n\left(\varphi_{n-1}^2+\varphi_{n+1}^2\right)\right]$$

2. For $2R >> l_0 \quad (\mu \to 0)$

$$\ddot{\phi}_n = -\frac{1}{12}\left[\left(\phi_{n-1}+\phi_n\right)^3 + \left(\phi_{n+1}+\phi_n\right)^3\right]$$

3. For $2R << l_0 \quad (\mu \to 1)$

$$\ddot{\varphi}_n = -\frac{1}{3}\varphi_n\left[\varphi_{n-1}^2 + 2\varphi_n^2 + \varphi_{n+1}^2\right]$$

Nonlinear dispersion

Consider a solution

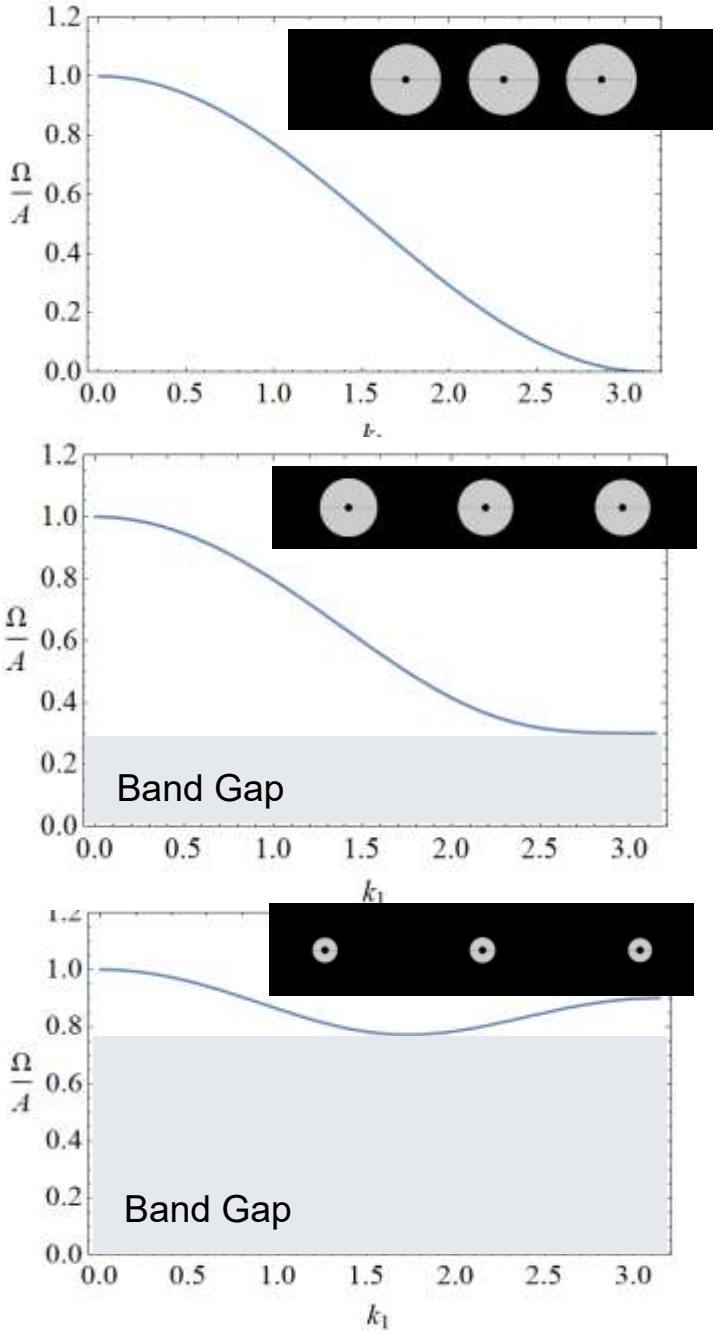
$$\varphi_n = \frac{1}{2} A_1 e^{i(\Omega\tau - k_1 n)} + \frac{1}{2} A_3 e^{i(3\Omega\tau - k_3 n)} + c.c.$$

Assumptions:

- RWA
- Dispersion ($A_3 << A_1$)

A. Self-propagation at ω

$$\Omega^2 = \frac{1-\mu}{8} |A_1|^2 \left(a + b + 4\cos k_1 + 2b\cos^2 k_1\right)$$



Nonlinear dispersion (3)

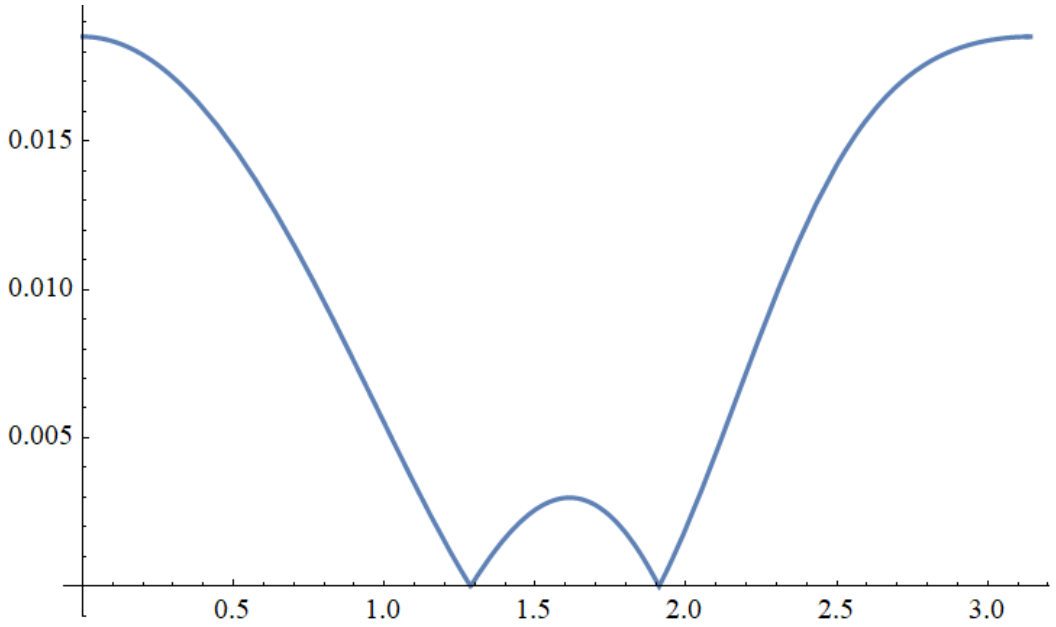
B. Propagation of 3ω, induced by pump at ω

$$(3\Omega)^2 = \frac{1-\mu}{8}|A_1|^2\left(a+b+2\cos k+2\cos k_3+2b\cos k\cos k_3\right)$$

Amplitude of third harmonic

$$A_3 = \frac{(1-\nu)^2}{12}\frac{A_1^3}{(3\Omega)^2}(1+a\cos k_1)\left(1-a\cos k_1-2\cos^2 k_1\right)$$

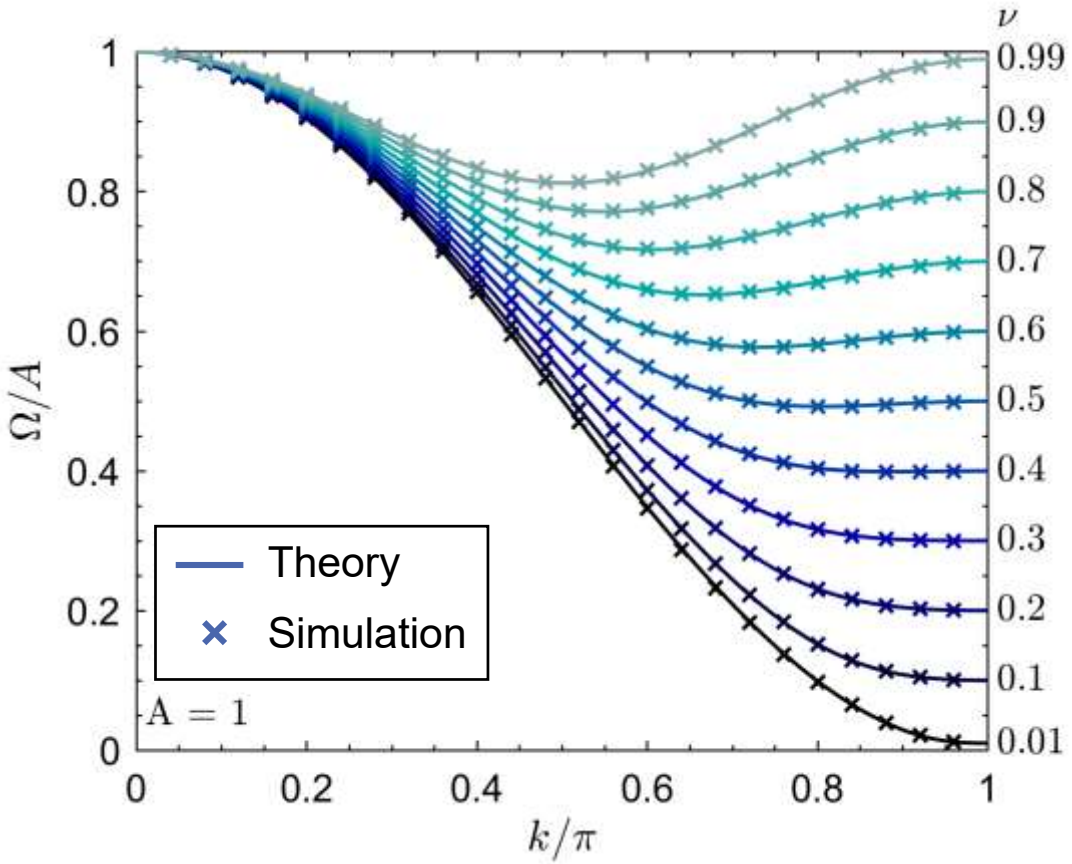
Amplitude vanishes at particular k: purely harmonic waves in purely nonlinear system!

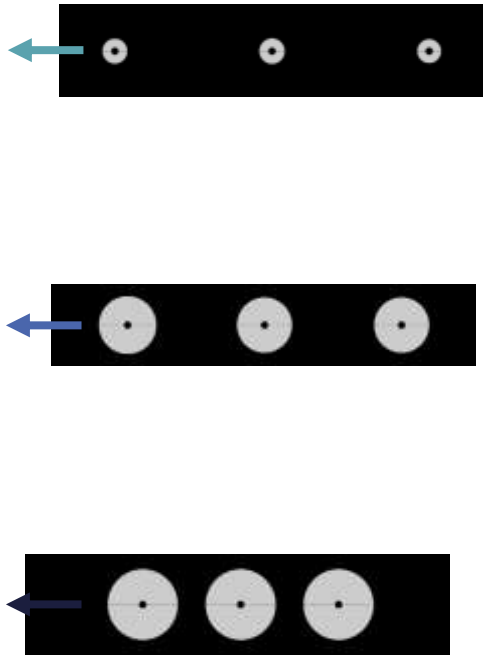


NUMERICAL

Dispersion relation

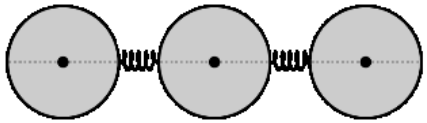
- We compute the space-time evolution of a given k-mode
- Transform to Ω-k space by FFT
- Short times: fine agreement



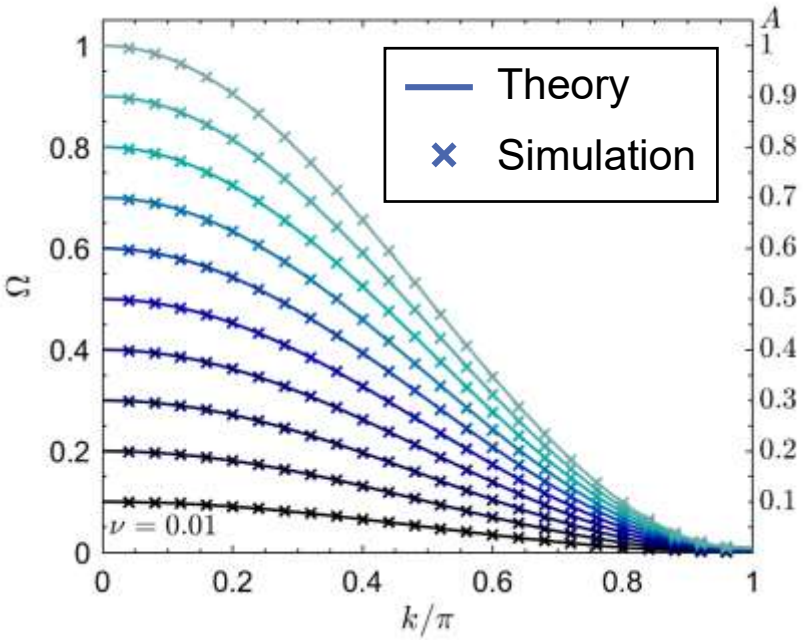


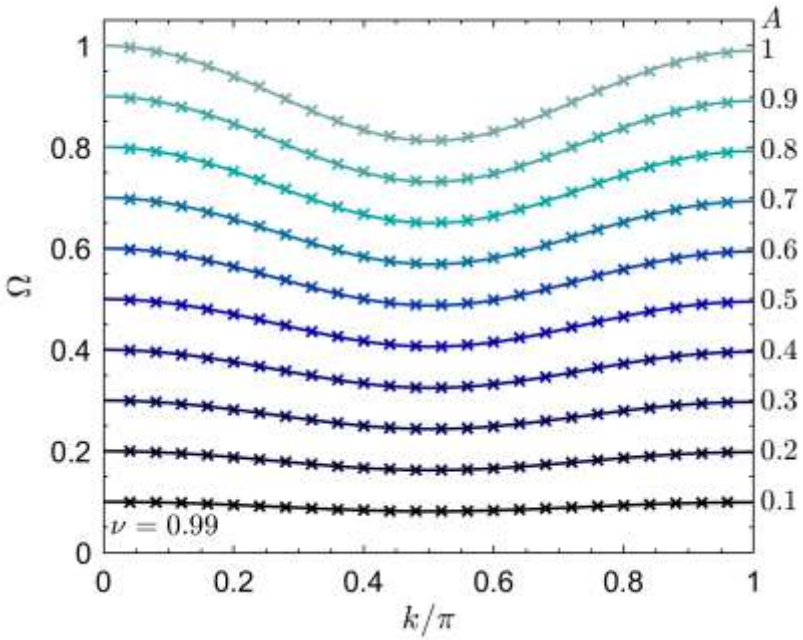
Dispersion relation

- Dispersion is amplitude dependent
- Numerical results fits theory for all amplitudes,
- For long times solution becomes compex…



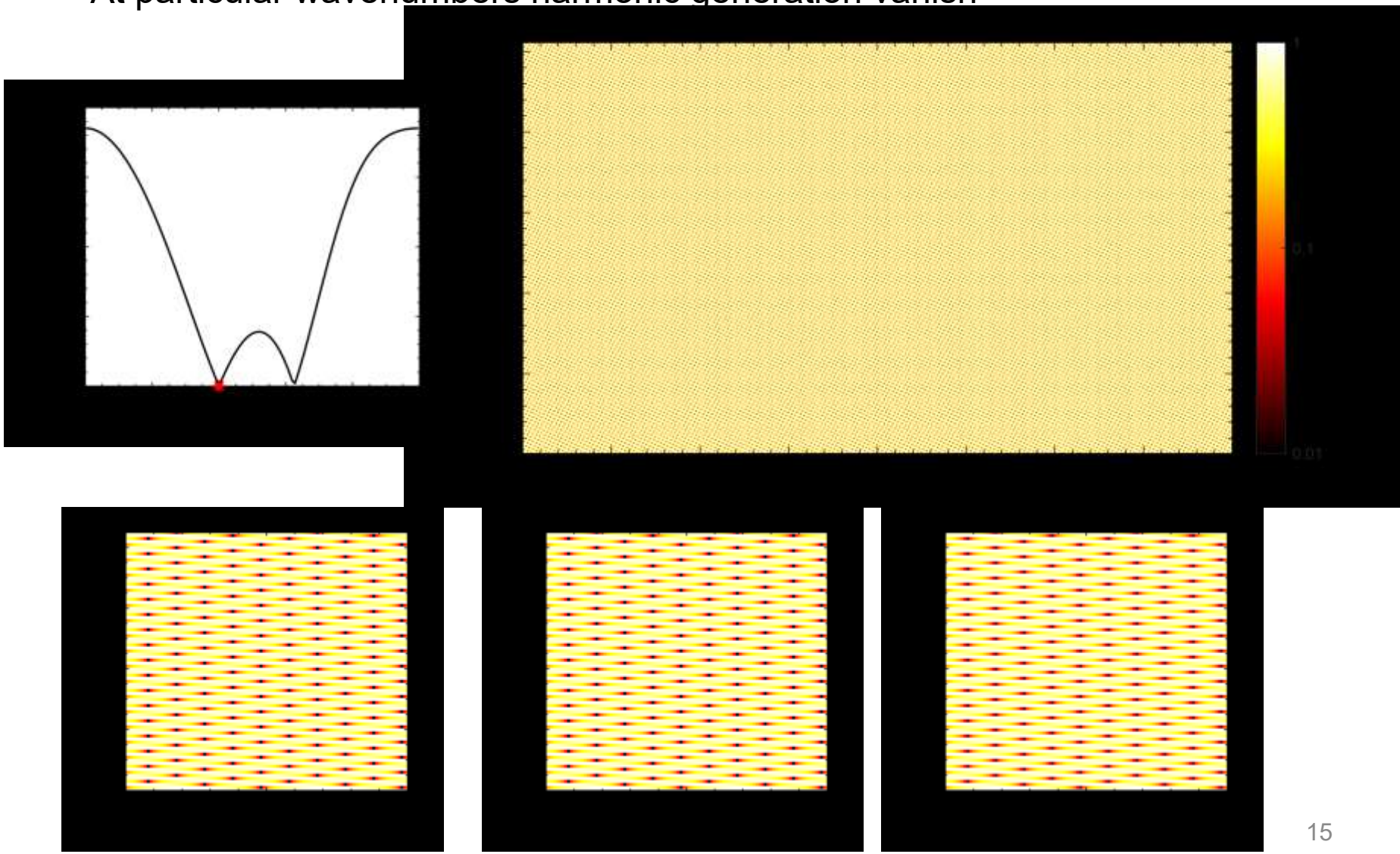




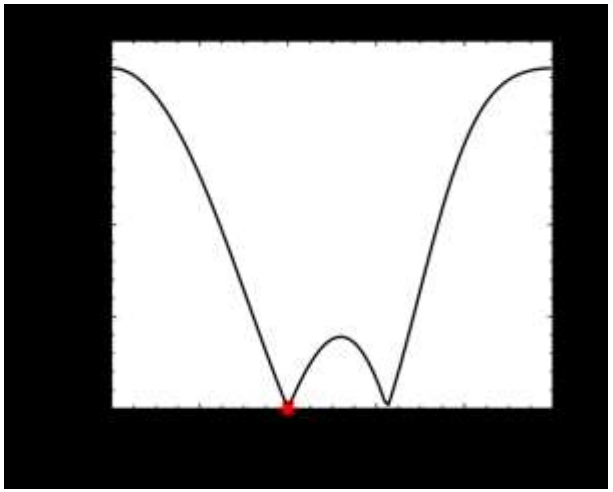


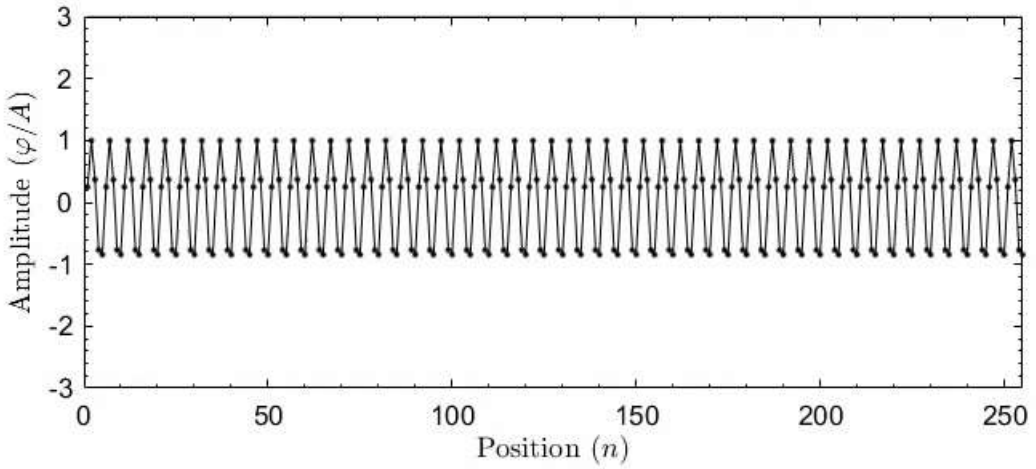
Magic wavenumber (1):

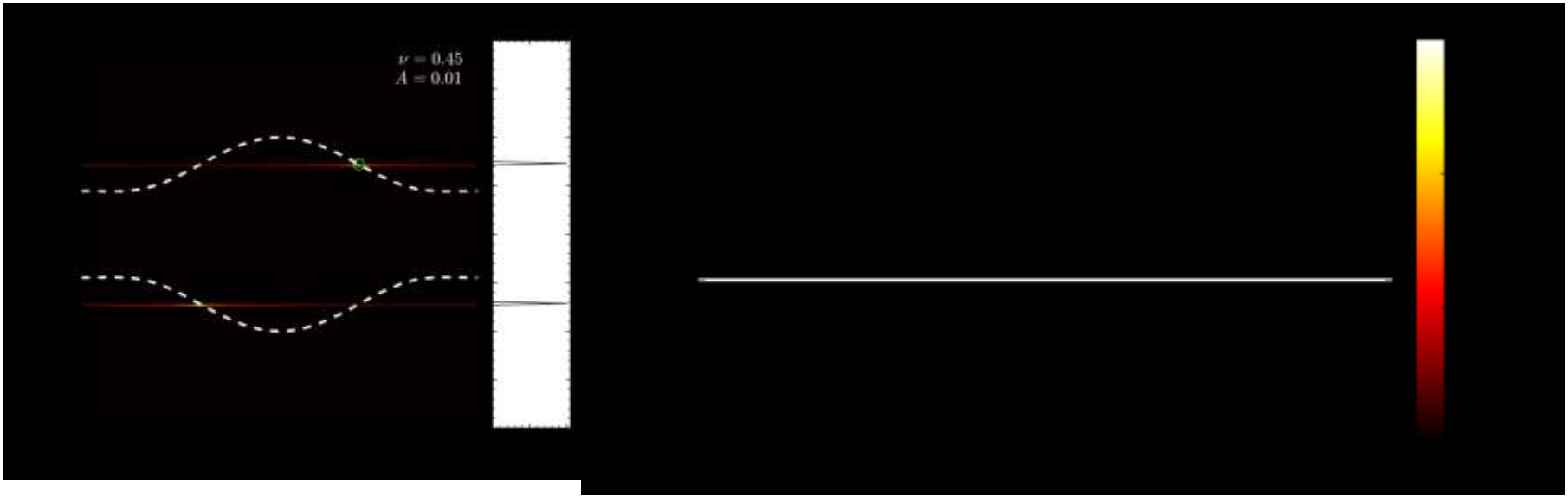
- At particular wavenumbers harmonic generation vanish



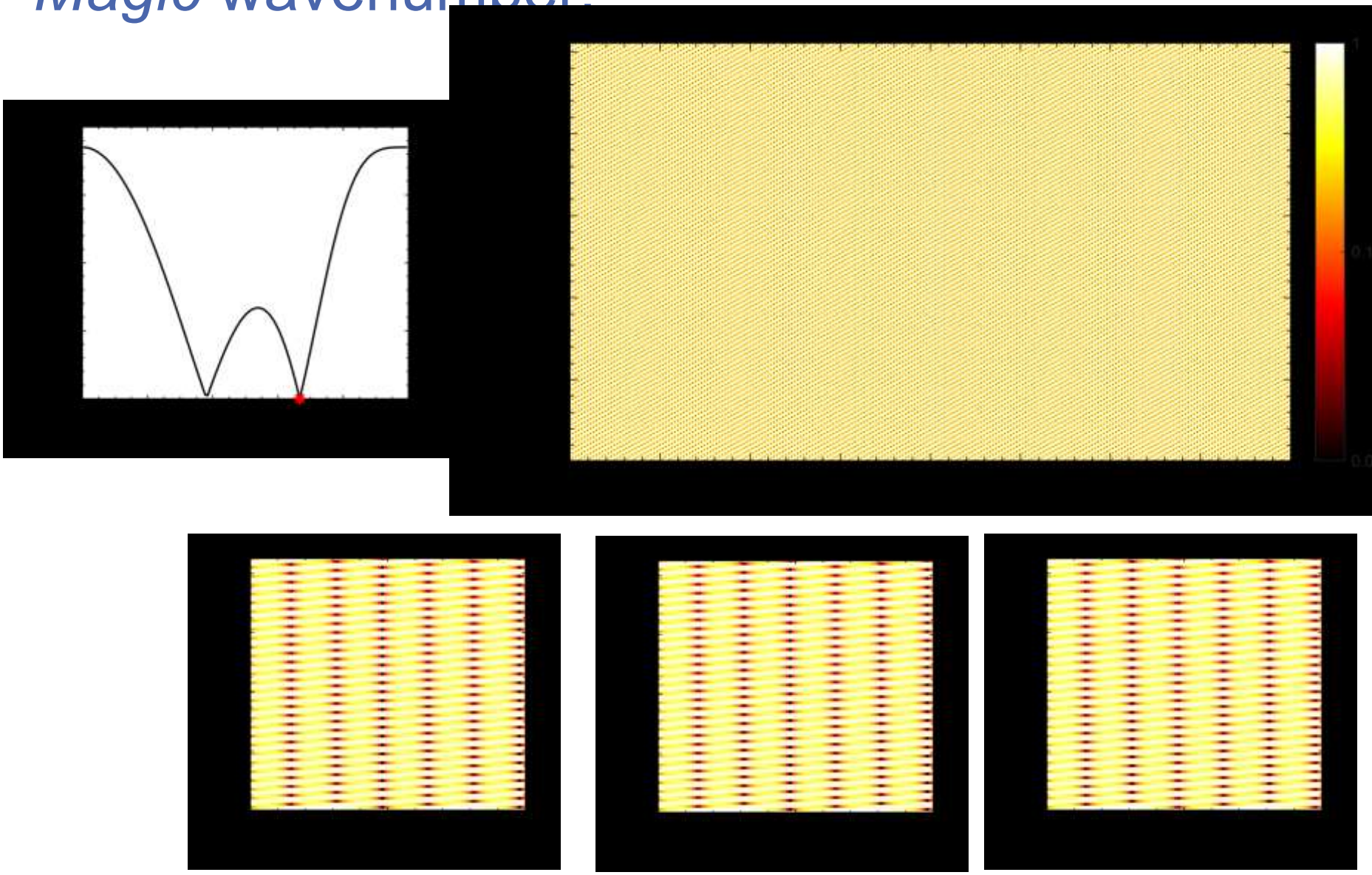
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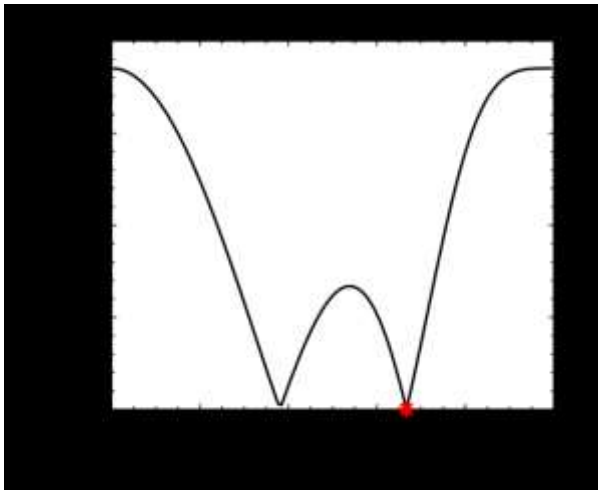


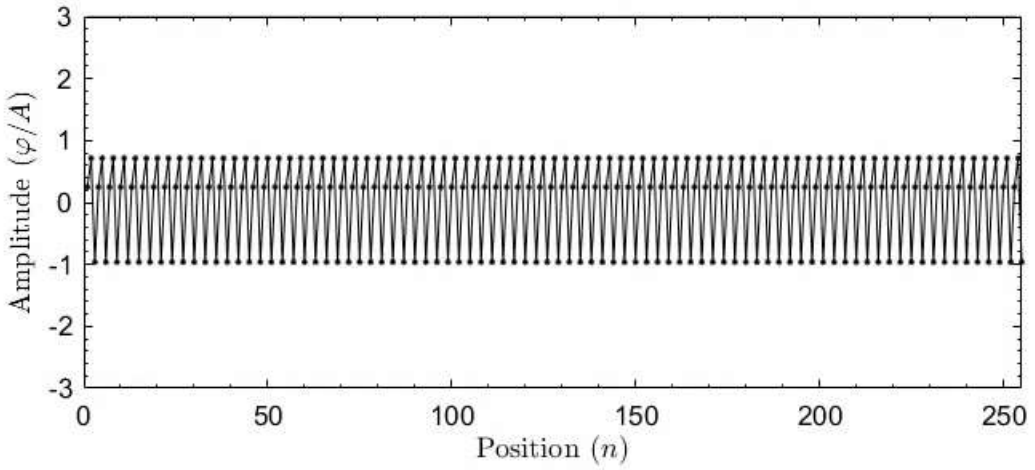


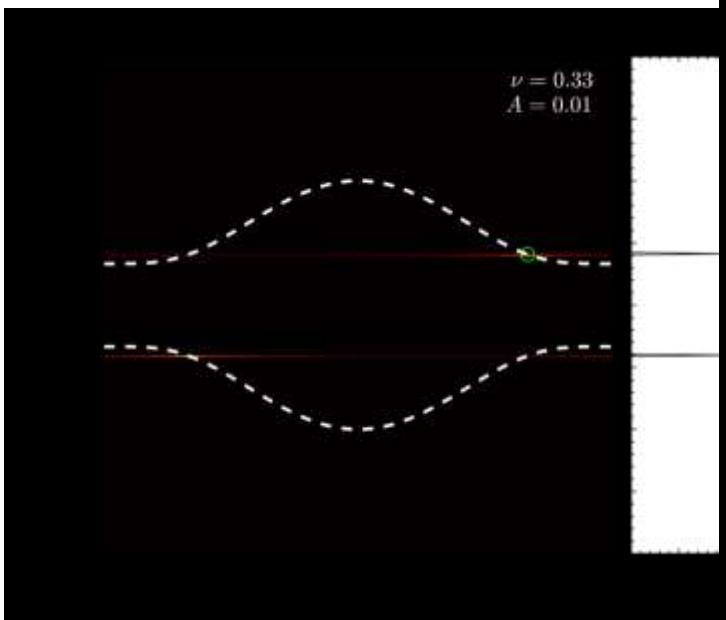
Magic wavenumber:

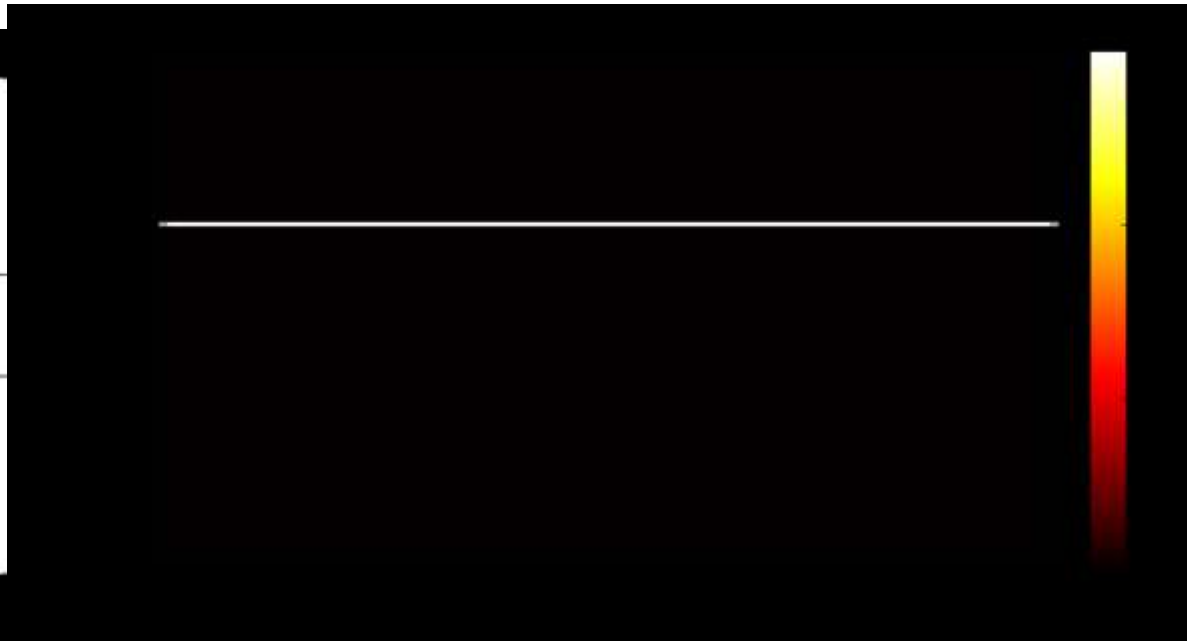


Magic wavenumber:

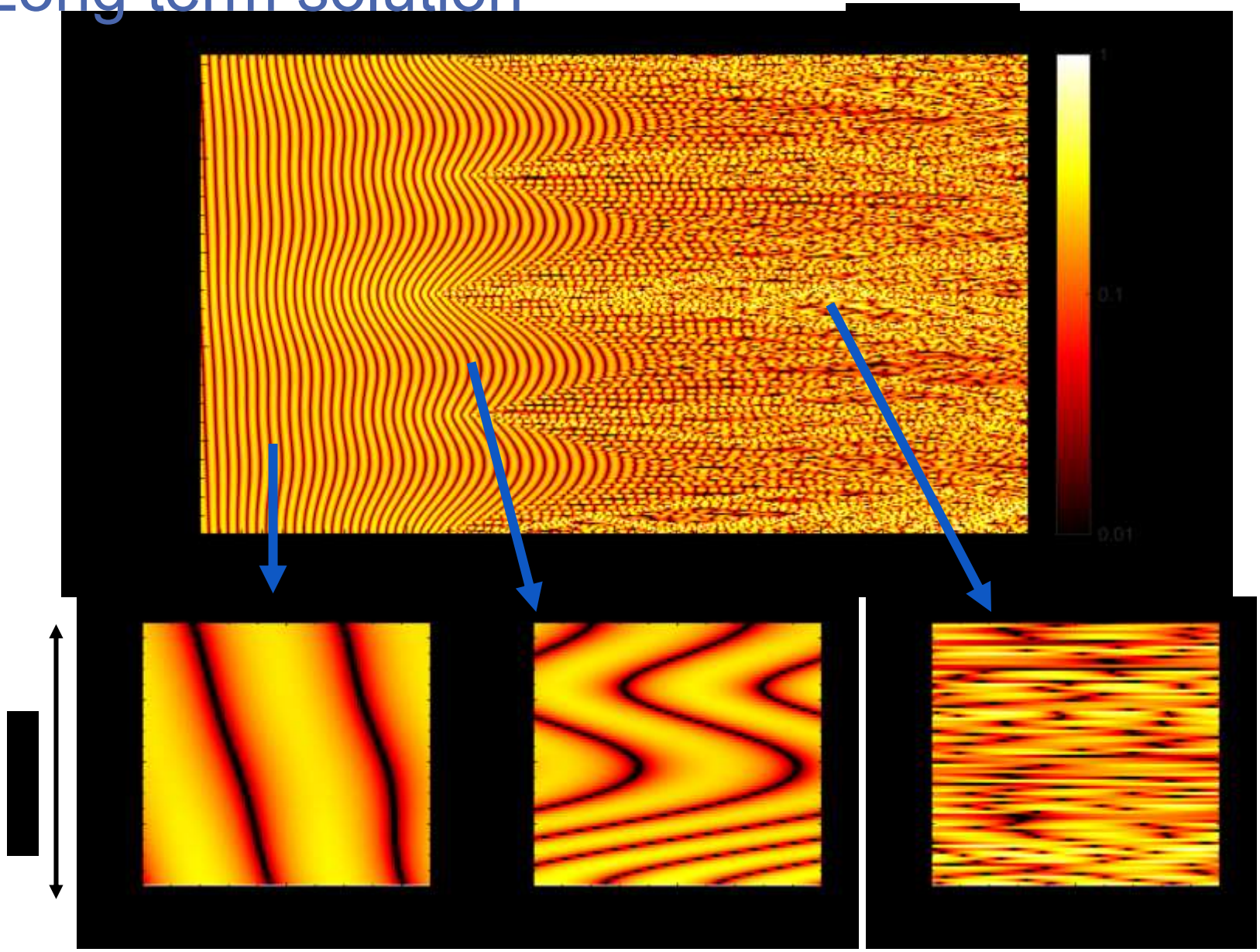




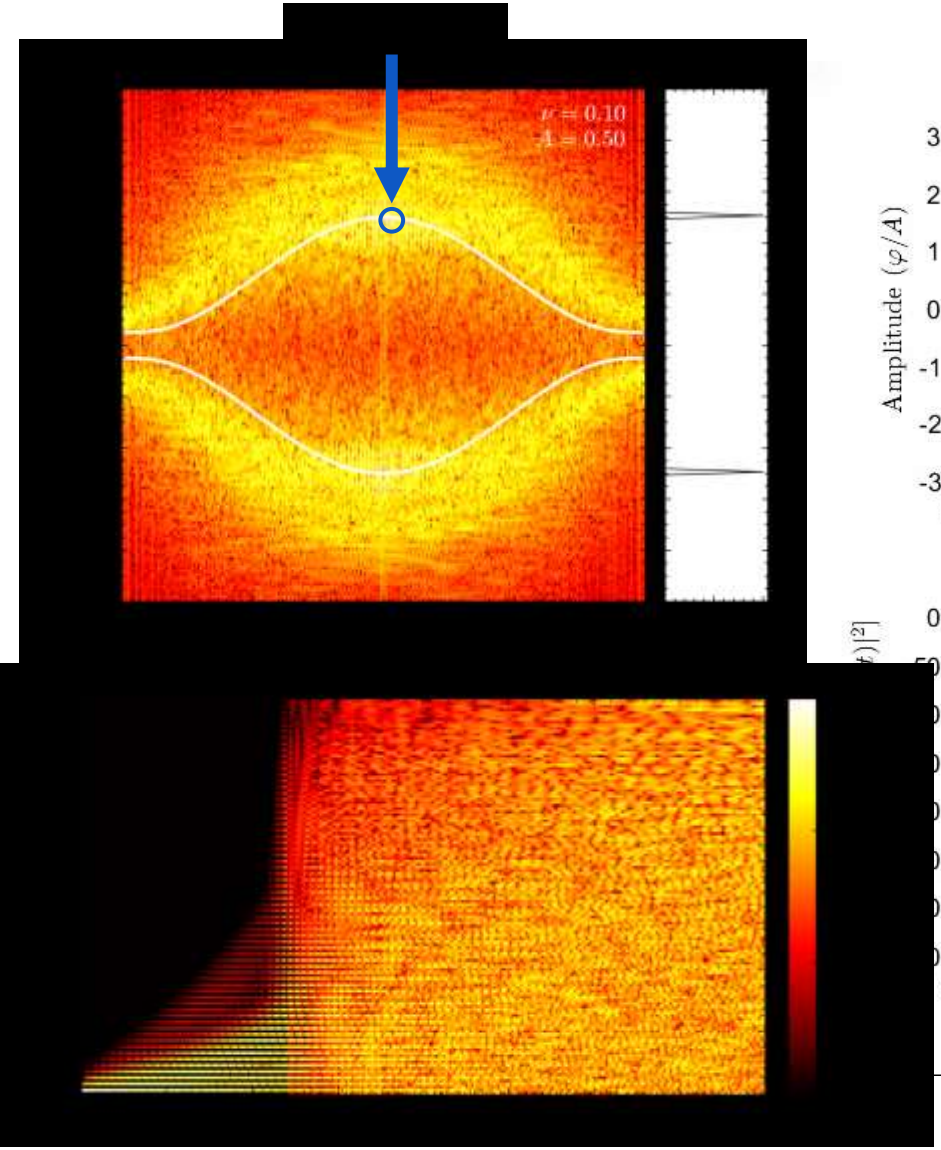


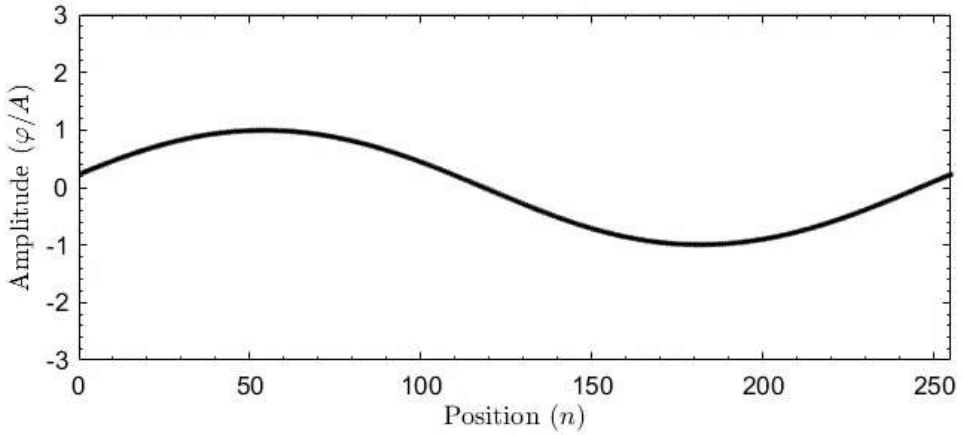


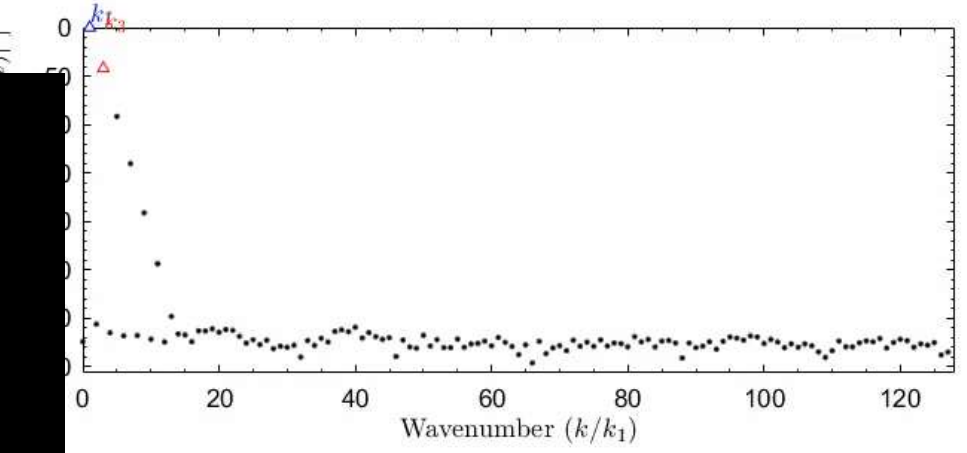
Long term solution



Long term solution







Compacton solutions

$$\ddot{\varphi}_n = -\frac{1}{12}\left[\left(\varphi_{n-1}+\varphi_n\right)^3+\left(\varphi_{n+1}+\varphi_n\right)^3\right]$$

- Consider a solution $\phi_n(t)=\varphi_n G(t)$

$$\frac{d^2G}{dt^2}+CG^3=0 \qquad G(t)=A\,cn(\omega t,k)$$ Jacobi elliptic function

$$\left[\left(\varphi_{n-1}+\varphi_n\right)^3+\left(\varphi_{n+1}+\varphi_n\right)^3\right]=C\varphi_n$$

$$\varphi_n=\cos\left[q(n-n_0)\right] \quad for \quad \left|q(n-n_0)\right|<\frac{\pi}{2}$$

$$\varphi_n=0 \quad otherwise$$

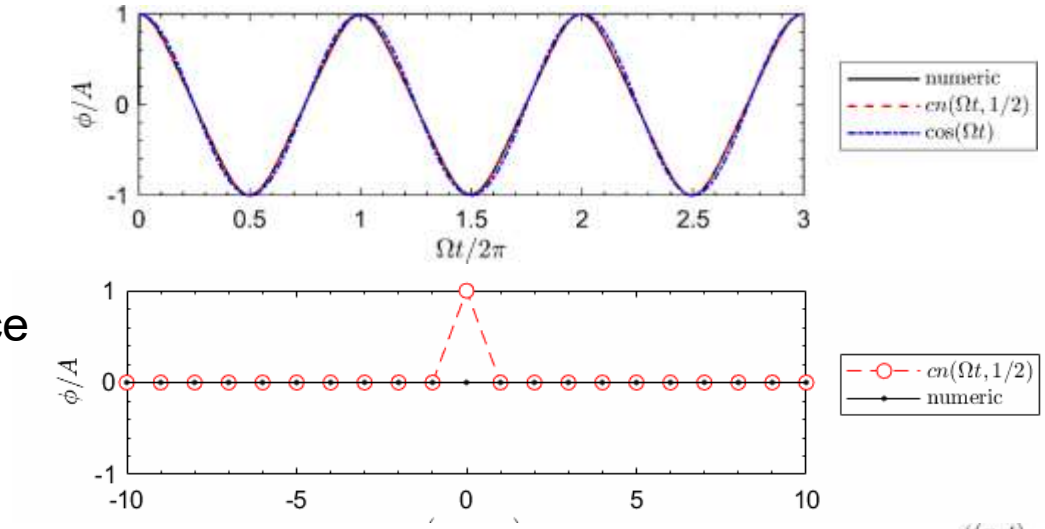
- Then

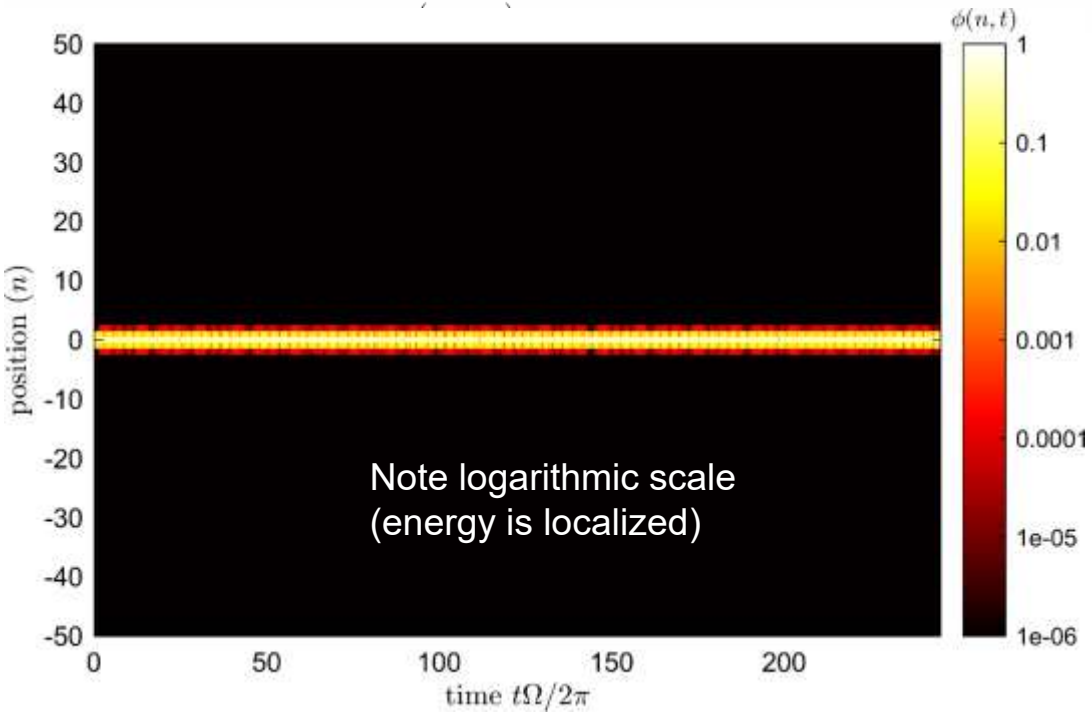
$$u_n(t)=A\cos\left[\frac{\pi}{3}(n-n_0)\right]cn\left(\omega t;\frac{1}{\sqrt{2}}\right) \quad for \quad \left|(n-n_0)\right|<\frac{3}{2}$$

$$u_n(t)=0 \quad otherwise$$

Compacton solutions

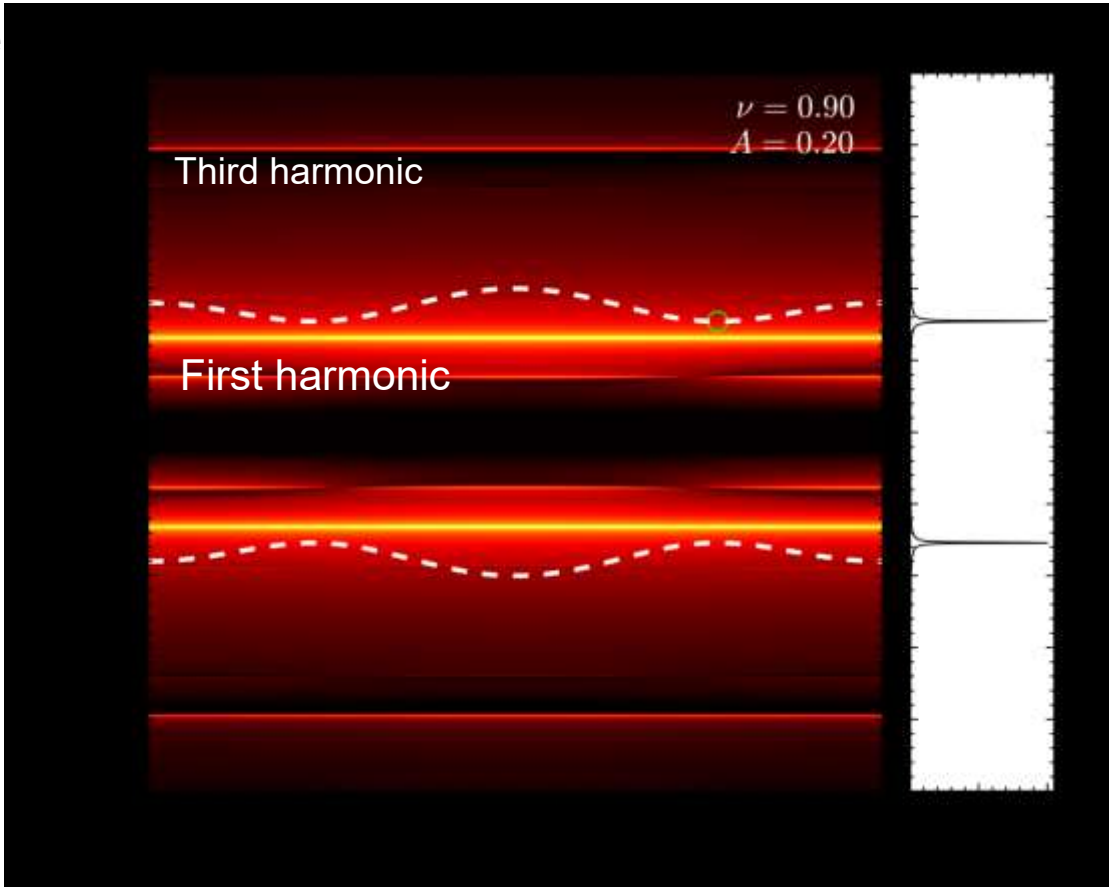
- Cnoidal solution (not harmonic)
- Periodic in time, compact in space





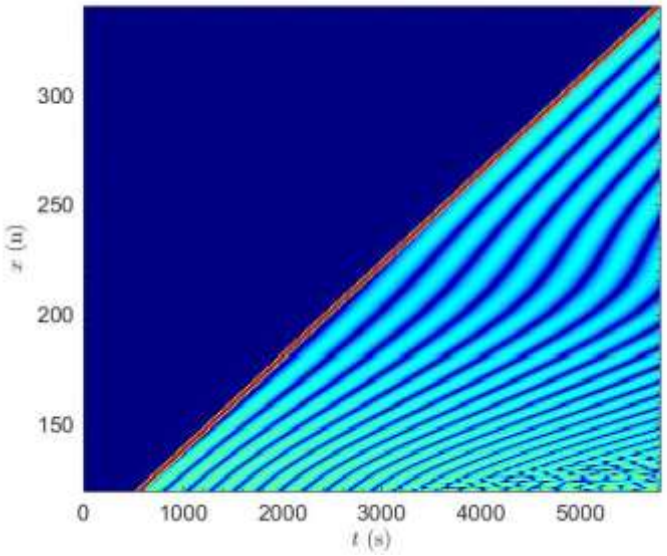
Compacton solutions

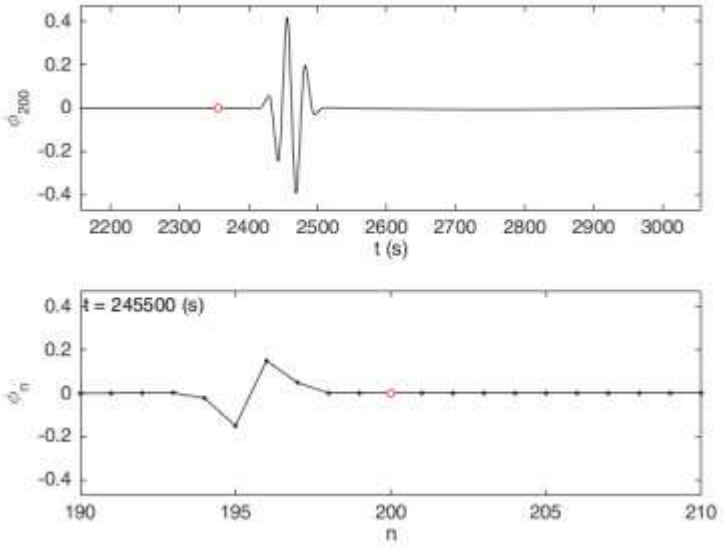
- Cnoidal solution (not harmonic)
- Periodic in time, compact in space



Dispersion, instability, solitons

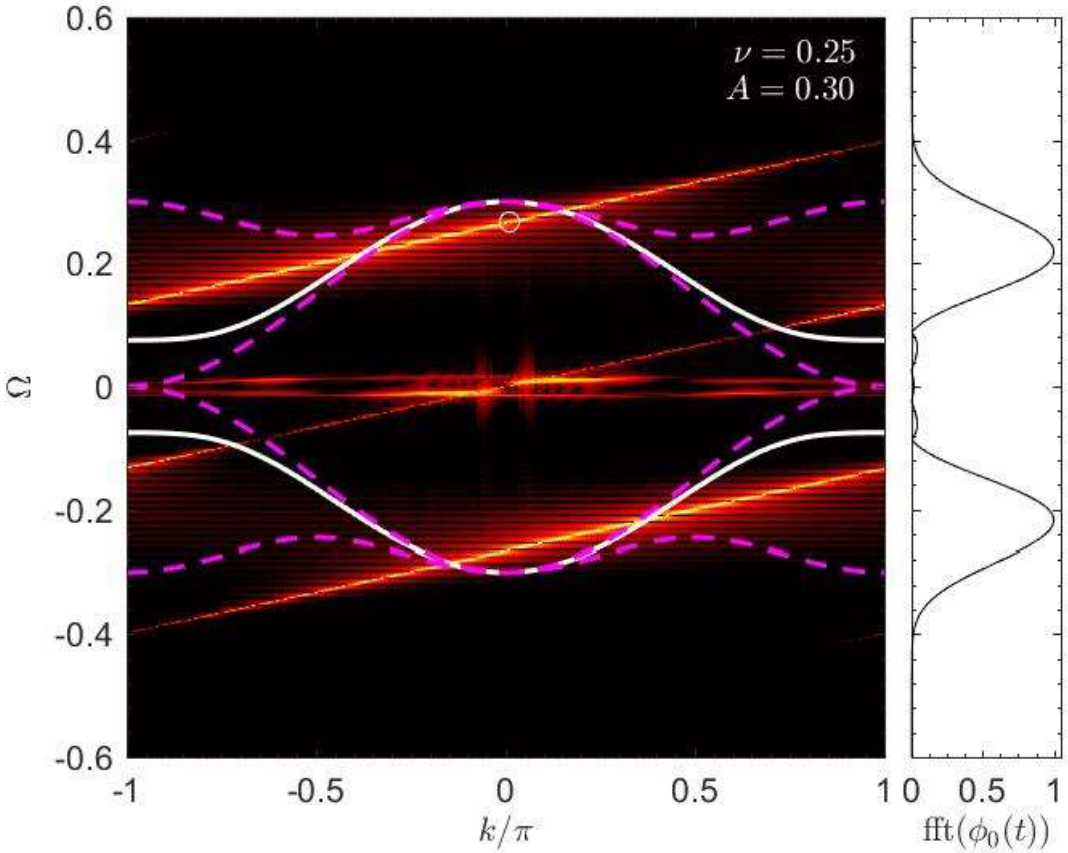
- We extract the profile of a compacton form the spontaneously generated sea of compactons
- We inject it as initial condition in the model





Dispersion, instability, solitons

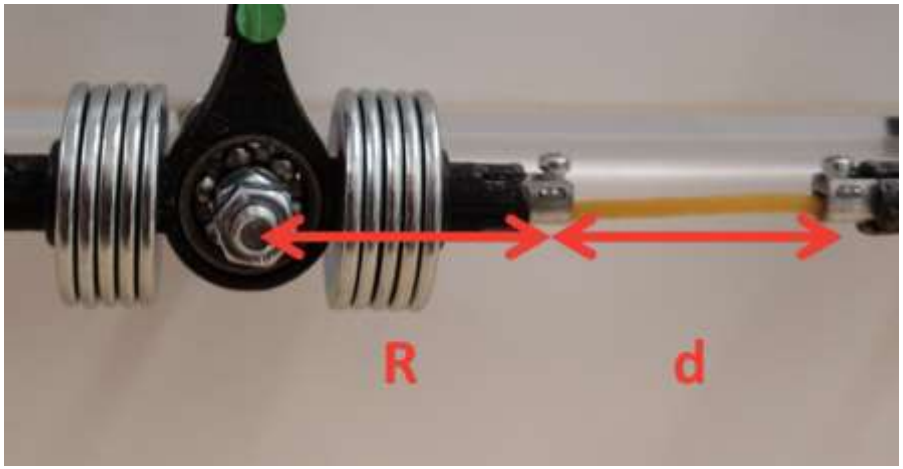
Long term evolution: straight lines - solitons



EXPERIMENT

Setup

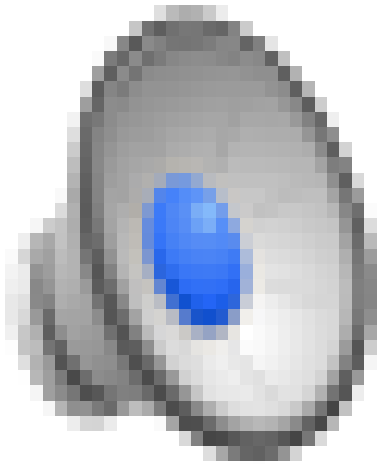






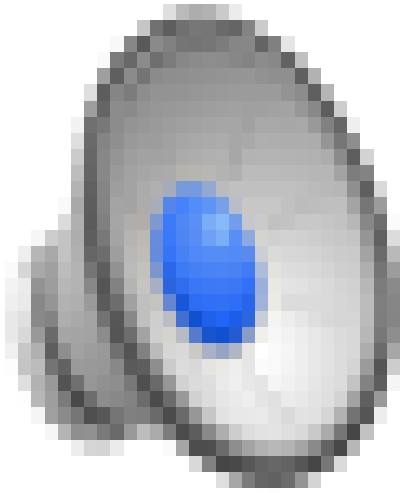
Experimental setup in action

- Waves in the upper bandgap



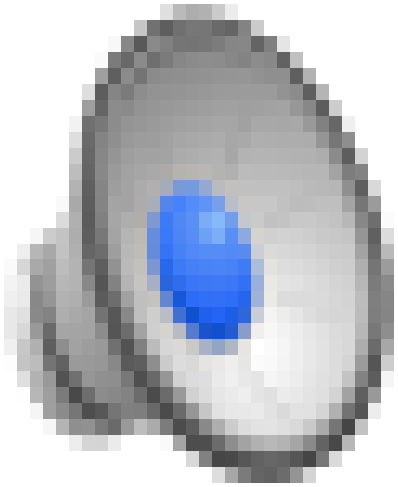
Experimental setup in action

- Waves in the propagation band

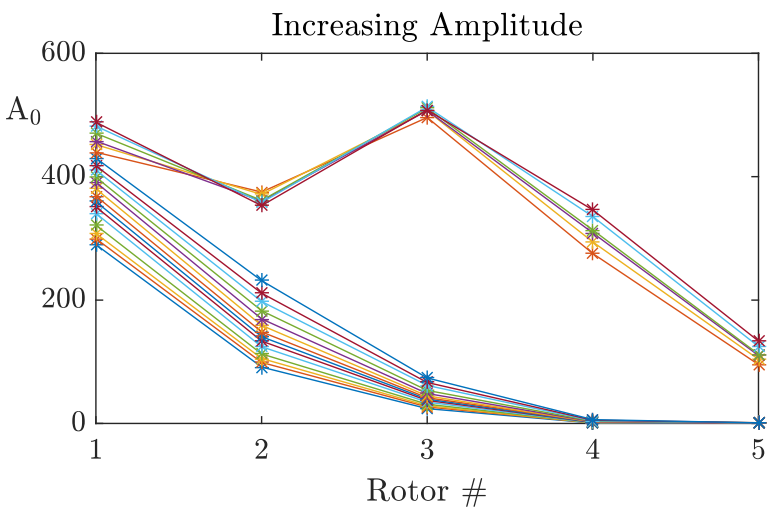


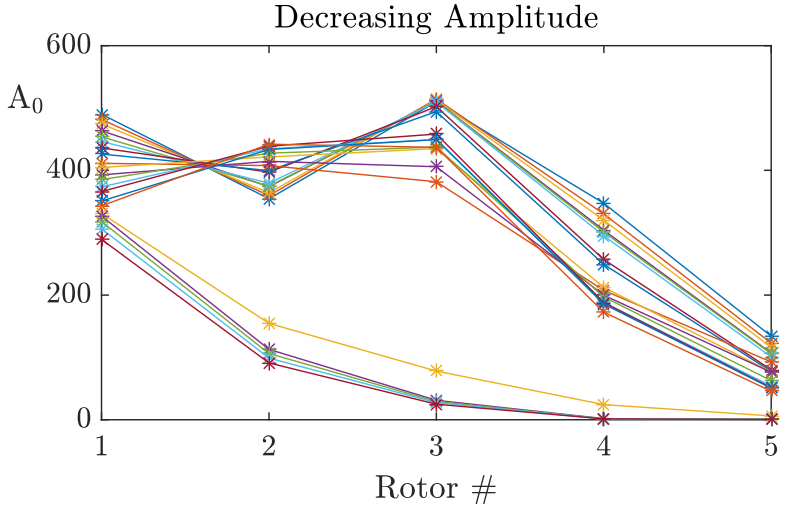
Experimental setup in action

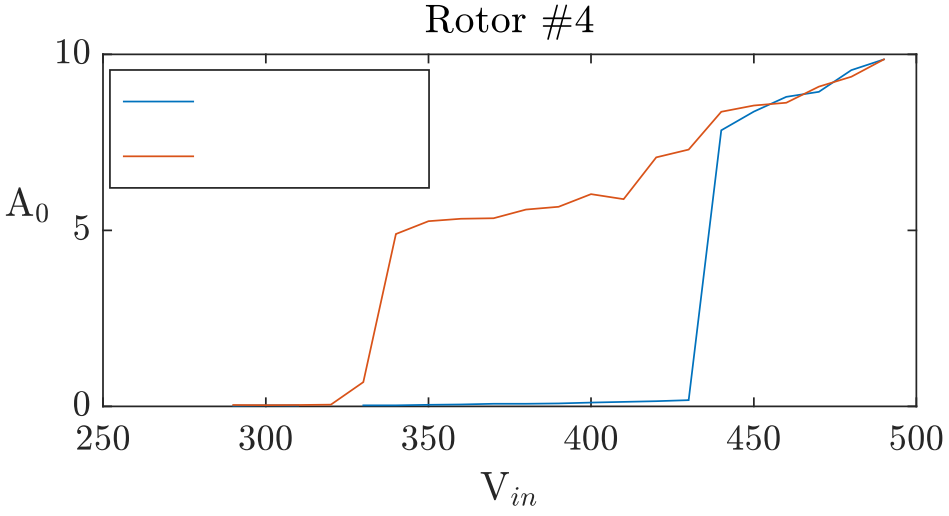
- Waves in the lower bandgap



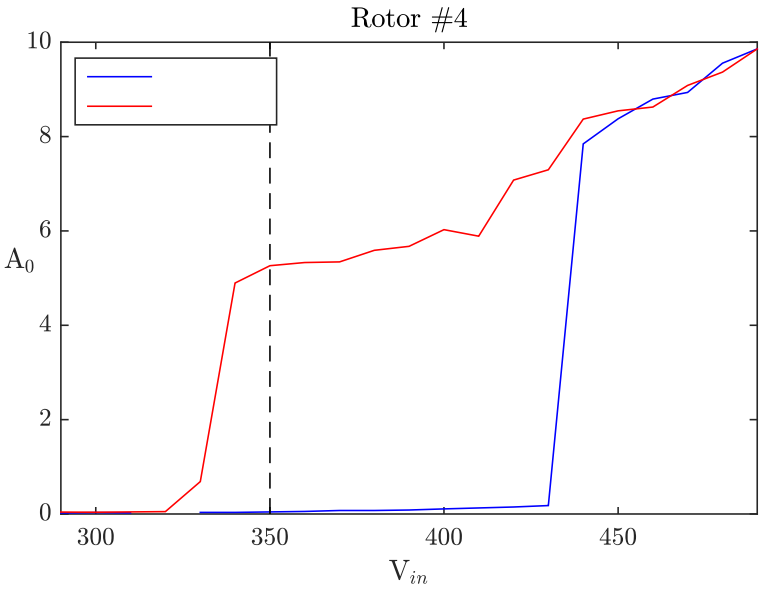
Hysteresis of evanescent waves

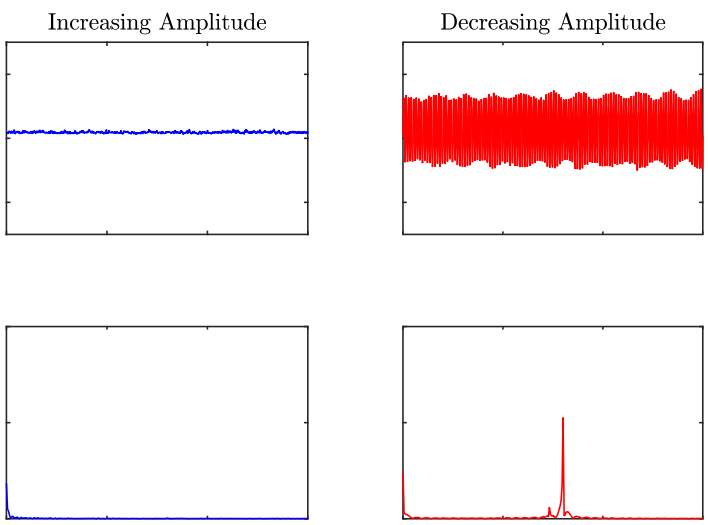




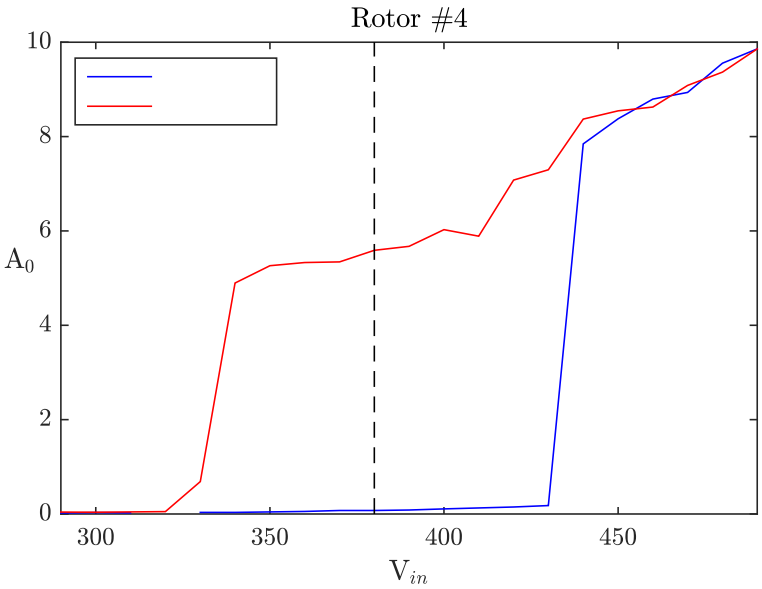


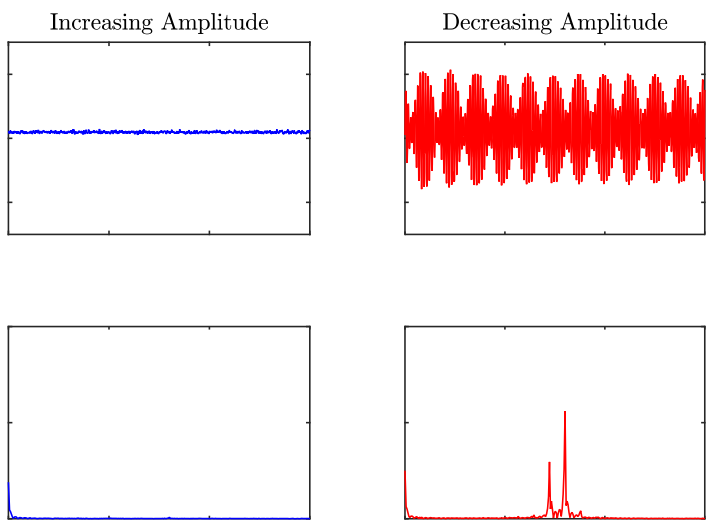
Modulational instability



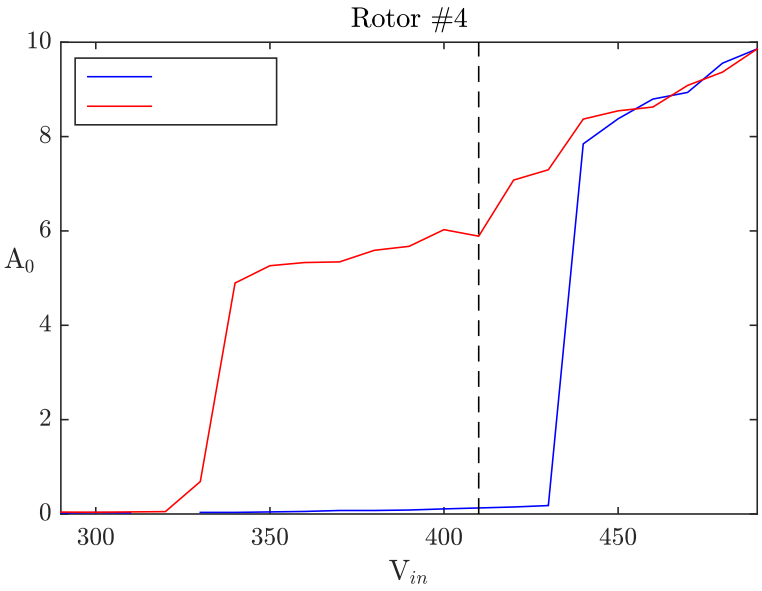


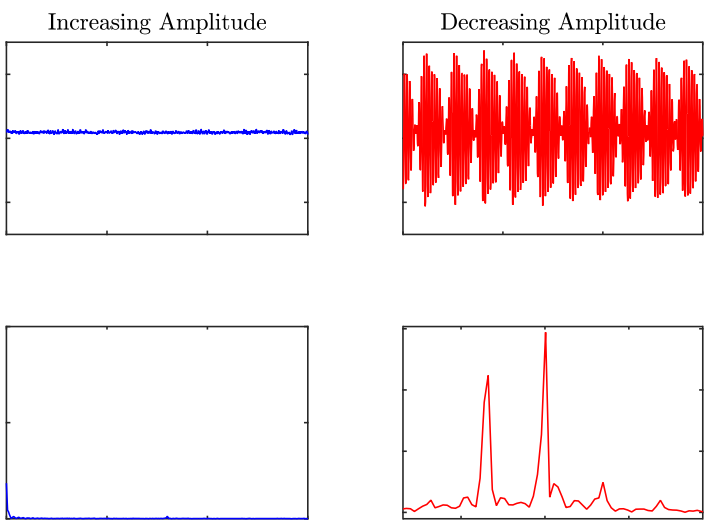
Modulational instability



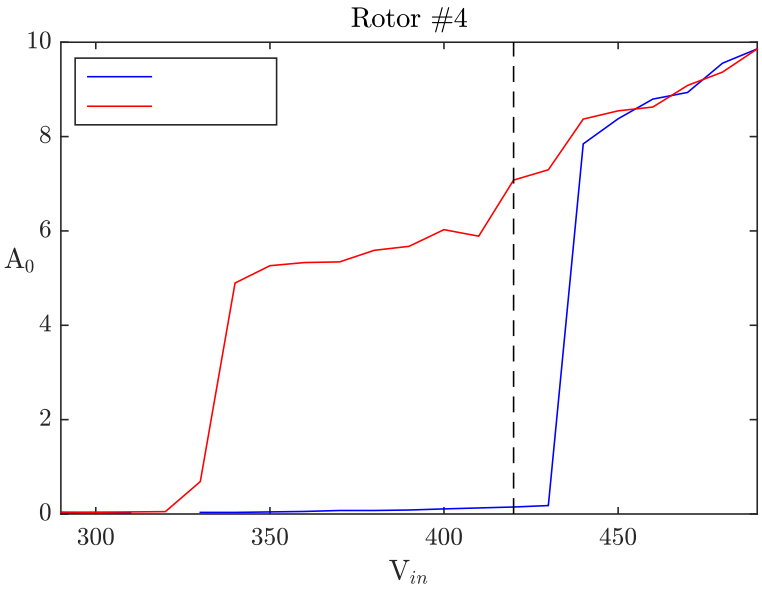


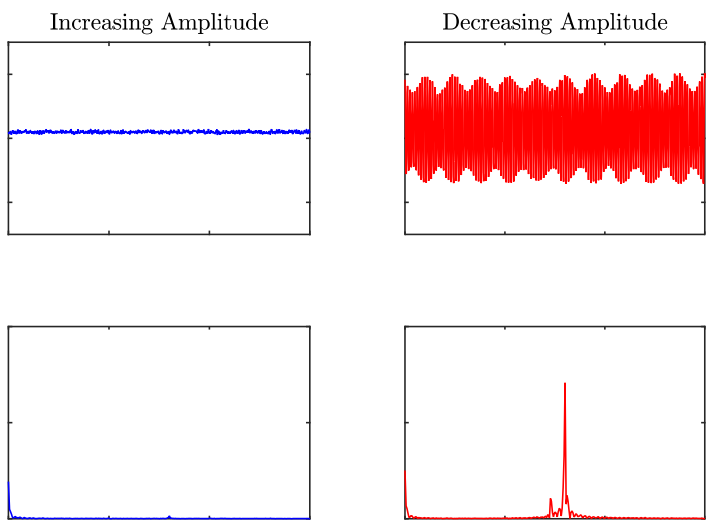
Modulational instability



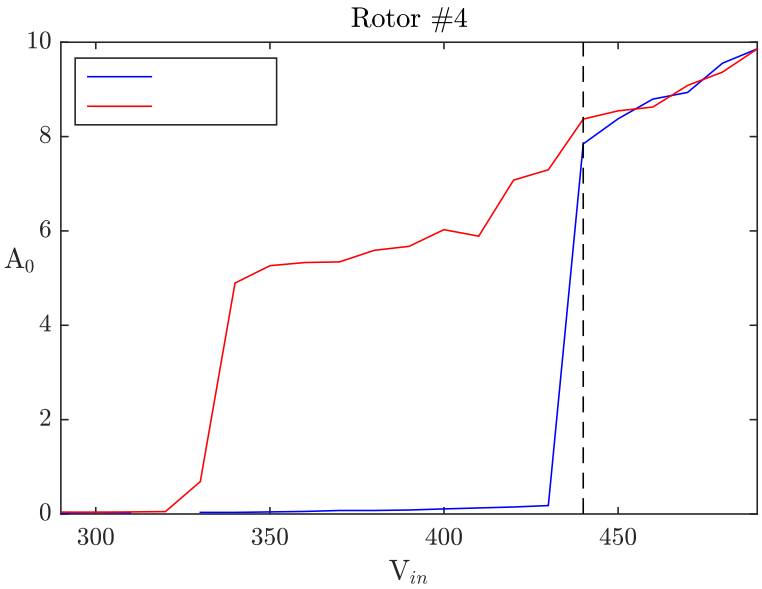


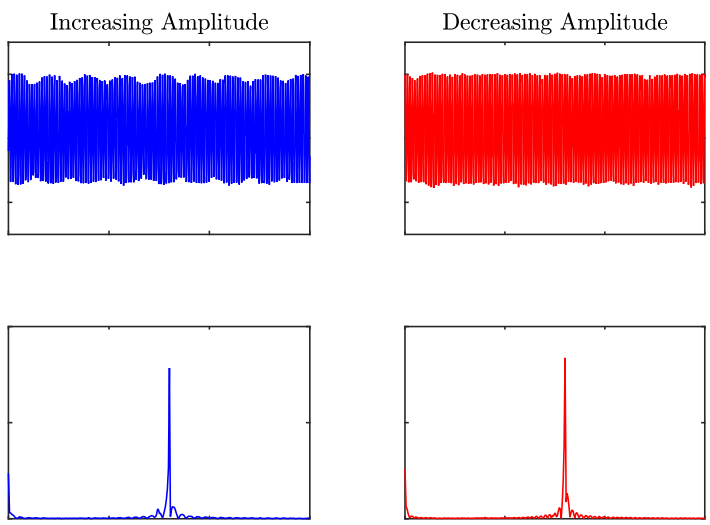
Modulational instability





Modulational instability





Conclusions

- A theoretical and experimental model for highly nonlinear waves
- Amplitude dependent dispersion
- Complex dynamics including
  - Hysteresis
  - Instabilities
  - Solitons
  - Spatio-temporal chaos

THANKS FOR YOUR ATTENTION